

Schwarz Symmetrization and Comparison Results for Nonlinear Elliptic Equations and Eigenvalue Problems

L. P. BONORINO¹ AND J. F. B. MONTENEGRO²

¹Departamento de Matemática Pura e Aplicada
Universidade Federal do Rio Grande do Sul
91509-900, Porto Alegre, RS, Brazil

²Departamento de Matemática, Universidade Federal do Ceará
60455-760, Fortaleza, CE, Brazil

Abstract

We compare the distribution function and the maximum of solutions of nonlinear elliptic equations defined in general domains with solutions of similar problems defined in a ball using Schwarz symmetrization. As an application, we prove the existence and bound of solutions for some nonlinear equation. Moreover, for some nonlinear problems, we show that if the first p -eigenvalue of a domain is big, the supremum of a solution related to this domain is close to zero. For that we obtain L^∞ estimates for solutions of nonlinear and eigenvalue problems in terms of other L^p norms.

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1 Introduction

In this work we study the L^p -norm and the distribution function of solutions to the Dirichlet Problem

$$\begin{cases} -\operatorname{div}(a(u, \nabla u)) &= f(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set in \mathbb{R}^n , $f : \mathbb{R} \rightarrow \mathbb{R}$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy some suitable conditions. First we assume the following hypotheses:

- (H1) f is a nonnegative locally Lipschitz function;
(H2) f is nondecreasing;
(H3) $a \in C^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}); \mathbb{R}^n)$ is given by $a(t, z) = e(t, |z|)z$, where $e \in C^1(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ is positive on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$, $a(t, 0) = 0$, $a(t, z) \cdot z$ is convex in the variable $z \in \mathbb{R}^n$ and $\partial_s(|a(t, sz)|) > 0$ for $z \neq 0$ and $s > 0$. Observe that the convexity of $z \mapsto a(t, z) \cdot z$ implies in the fact that $s \mapsto |a(t, sz)|$ is increasing.
(H4) there exist $p \geq q > 1$, $q_0 > 1$, and positive constants C_s , C_* and C^* s.t.

$$C_s |z|^{q_0} \leq \langle a(t, z), z \rangle \quad \text{for } |z| \leq 1, t \in \mathbb{R}$$

and

$$C_* |z|^q \leq \langle a(t, z), z \rangle \leq C^* (|z|^p + |t|^p + 1) \quad \text{for } |z| \geq 1, t \in \mathbb{R}.$$

Hence, using that $s \mapsto a(t, sz) \cdot sz$ is increasing and positive,

$$C_* (|z|^q - 1) \leq a(t, z) \cdot z \leq C^* (|z|^p + |t|^p + 1) \quad \text{for } z \in \mathbb{R}^n$$

and

$$C_* (\lambda_B \|w\|_q^q - |\Omega|) \leq \int_{\Omega} a(w, \nabla w) \cdot \nabla w \, dx \leq C^* (\|\nabla w\|_p^p + \|w\|_p^p + |\Omega|), \quad (1.1)$$

for $w \in W_0^{1,p}(\Omega)$, where λ_B is the first eigenvalue of $-\Delta_q$ in a ball B , that has the same measure as Ω .

(H5) there exist $\beta \geq 0$ and $\alpha < C_* \lambda_B$ such that

$$0 < f(t) \leq \alpha t^{q-1} + \beta \quad \text{for } t > 0.$$

At first our main concern is to compare the maximum and the distribution function of a solution associated to Ω with one associated to B . We can obtain even a priori estimates of solutions for some problems with nonlinear lower order terms and prove the existence of solution. Later on we see also some applications for these estimates, including L^∞ estimates for some eigenvalue and nonlinear problems. So we show that if a domain is “far away” from the ball (ie, its first p -eigenvalue is big), then the maximum of a solution is small. Indeed the supremum of a solution is bounded by some negative power of the first p -eigenvalue. This kind of question seems to be new and the works in the literature normally are focused in comparing solutions with a radial one, disregarding better estimates when the domain is not close to a ball.

More precisely, let B be the open ball in \mathbb{R}^n , centered at the origin, such that $|B| = |\Omega|$, where $|C|$ denotes the Lebesgue’s measure in \mathbb{R}^n of a measurable set C , and consider the function U_B given by

$$U_B(x) = \sup\{U(x) \mid U \in W_0^{1,p}(B) \text{ is a radial solution of } (\tilde{P}_B)\}, \quad (1.2)$$

where (\tilde{P}_B) is the Dirichlet Problem

$$\begin{cases} -\operatorname{div}(\tilde{a}(U, \nabla U)) &= f(U) & \text{in } B \\ U &= 0 & \text{on } \partial B. \end{cases} \quad (\tilde{P}_B)$$

Let u be a weak solution of

$$\begin{cases} -\operatorname{div}(a(v, \nabla v)) &= f(v) & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega. \end{cases} \quad (P_\Omega)$$

in $W_0^{1,p}(\Omega)$. Observe that u and U_B are positive. Define the distribution function of u by

$$\mu_u(t) = |\{x \in \Omega : u(x) > t\}|.$$

For a , \tilde{a} and f satisfying hypotheses (H1)-(H5) (the constants and powers related to a and \tilde{a} can be different) and $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$, we prove that U_B is a solution of (\tilde{P}_B) and, in Theorem 5.1,

$$\mu_u(t) \leq \mu_B(t), \quad \forall t \in [0, \max U_B], \quad (1.3)$$

where μ_B is the distribution function of U_B . If Ω is not a ball, $a = a(z)$ and $(a(z) \cdot z)^{1/p}$ is convex, then this inequality is strict.

We also prove some sort of maximum principle with respect to the solutions in the ball in the following sense: if u and U are solutions of (P_Ω) and (\tilde{P}_B) respectively, $u^\# \leq U$ (not necessarily maximal solution) and $u^\# \neq U$, then $u^\# < U$ provided f and a satisfy suitable conditions.

As an application we obtain this comparison to the problem with lower order terms

$$\begin{cases} -\operatorname{div}(a(\nabla u)) - \frac{h'(u)}{h(u)} \nabla u \cdot a(\nabla u) &= g(u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $h \in C^1$ is bigger than some positive constant, $f = gh$ and $a_1(t, z) = h(t)a(z)$ satisfy (H1)-(H5). This holds even if h has a bad growth and does not satisfy the upper inequality of (1.1). For the special case

$$-\Delta_p u - \frac{h'(u)}{h(u)} |\nabla u|^p = g(u)$$

this priori estimate can be used to prove existence of solution.

Moreover we get also some result even when f is not nondecreasing. Indeed, if f is positive, $f(t)/t^{p-1}$ is decreasing and $a(t, z) = \tilde{a}(t, z) = z|z|^{p-1}$, we show that

$$\max U_B \geq \max u.$$

This L^∞ estimate can be easily extended to the problem

$$\begin{cases} -\Delta_p v + k(v) &= f(v) & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

where k is positive and nondecreasing and f is positive and $f(t)/t^{p-1}$ is decreasing.

Then we apply these results to prove that if $w \in W_0^{1,p}(\Omega)$ is a solution of $\operatorname{div}(a(x, \nabla w)) = f(w)$ in Ω , where a satisfies some conditions and $f \in C^1(\mathbb{R})$ is bounded by $c|t|^{q-1} + d$, with $1 < q \leq p$ and $c, d \geq 0$, then

$$\|w\|_\infty \leq C_1 \|w\|_r^{\frac{rp}{n(p-q)+rp}} + C_2 \|w\|_r^{\frac{rp}{n(p-1)+rp}},$$

where $C_1 = C_1(n, p, q, r, \rho, c)$ and $C_2 = C_2(n, p, r, \rho, d)$ are positive constants. In the special case $|\Delta_p w| \leq |\lambda| |w|^{q-1}$, where $\lambda \in \mathbb{R}$, we have

$$\|w\|_s \leq \left[\frac{2}{(\omega_n)^{1/r}} \left(\frac{2(p-1)}{p} \right)^{\frac{n(p-1)}{rp}} \left(\frac{|\lambda|}{n} \right)^{n/rp} \right]^{\frac{s-r}{\kappa s}} \|w\|_r^{\frac{s-r}{\kappa s} + \frac{r}{s}}, \quad (1.6)$$

where $0 < r < s$ and $\kappa = 1 + \frac{n(p-q)}{rp}$. These inequalities imply, according to Corollary 7.1, in a L^∞ -norm decay of the solutions of some sublinear equations, when the domain becomes “far away” from a ball with the same volume. Since the ball is the domain of a given measure that maximizes the L^p norms in several problems, it would be interesting to obtain better estimates for solutions that are not defined in a ball. Hence, we need to measure in some way the difference between its domain and the corresponding ball. The first eigenvalue is a possible form of distinction between these sets, that we use to establish some upper bound. Finally, as an application, we prove that $u^\sharp < U$, where u is a solution of (P_Ω) and U a solution of (\tilde{P}_B) , even when f is not monotone, provided the first eigenvalue associated to Ω , $\lambda_p(\Omega)$, is big enough and some conditions on a and f are satisfied.

We point out that we are not interested in establishing existence of solutions for (P_Ω) . Our main concern is just to compare these solutions and we obtain existence results only for the radial case.

Results of this type have been obtained by several authors. In [42] Talenti proved that if u is the weak solution of the Dirichlet Problem

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u = f(x) \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega,$$

where $c(x) \geq 0$, $\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \xi_1^2 + \dots + \xi_n^2$ and v is the weak solution of

$$-\Delta v = f^\sharp \text{ in } B \quad \text{and} \quad v = 0 \text{ on } \partial B,$$

where B is the ball centered at 0 such that $|B| = |\Omega|$ and f^\sharp is the decreasing spherical rearrangement of f , then $\text{ess sup } u \leq \text{ess sup } v$ and $\mu_u \leq \mu_v$. As a consequence, $\|v\|_{L^p}/\|f^\sharp\|_{L^q} \geq \|u\|_{L^p}/\|f\|_{L^q}$. This estimate is an extension of the one previously obtained by Weinberger [46] for the ratio $\|u\|_{L^\infty}/\|f\|_{L^q}$. Further results have been proved for a larger class of linear equations that either satisfy weaker ellipticity conditions (see [8], [9]) or contain lower order terms (see [4], [6], [7], [10], [18], [27], [44], [45]). Similar problems were studied in [34], [35], [36].

As in the linear case, estimates have been obtained for solutions $u \in W_0^{1,p}(\Omega)$ to the nonlinear problem

$$-\sum_{i=1}^n (a_i(x, u, \nabla u))_{x_i} - \sum_{i=1}^n (b_i(x) |u|^{p-2} u)_{x_i} + h(x, u) = f(x, u) \text{ in } \Omega,$$

comparing the decreasing spherical rearrangement of u with the solution of some nonlinear “symmetrized” problem. For instance, the case $b_i = h = 0$ and $\sum a_i(x, u, \xi) \xi_i \geq A(|\xi|)$, where A is convex and $\lim_{r \rightarrow 0} A(r)/r = 0$, is considered in [43]. The problem in a general form is studied in [14], assuming that the coefficients are in suitable spaces and $\sum a_i(x, u, \xi) \xi_i \geq |\xi|^p$. Under similar hypotheses, the case $b_i = 0$ is considered in [25] and different comparison results are obtained. In [2] estimates are proved when the coefficients satisfy $b_i = h = 0$, $a_i = a_i(Du)$ and $\sum a_i(\xi) \xi_i \geq (H(\xi))^2$, where H is a nonnegative convex function, positively homogeneous of degree 1. Other related result were established in [1], [26], [39]. Some results also extend to parabolic equations (see e.g. [2], [6], [12]).

Usually comparison results are obtained considering a “symmetrized equation” that is different from the original one. In this work we can keep the original equation and symmetrize only the domain, obtaining sharper estimates. Results similar to ours are established in [11], [37] for the laplacian operator, where the authors apply the method of subsolution and supersolution to prove that, for a given symmetric solution U in the ball, there exists some solution in Ω for which the symmetrization is less than U . Indeed, applying the iteration procedure used in those works and the main result of [43], the estimate (1.3) can be obtained in the particular case $-\text{div}(a(\nabla u)) = f(u)$, provided we have some a priori estimate in the L_q norm for subsolutions and the existence of the maximal radial solution U_B . Using different techniques, we prove in Section 5 that the symmetrization of any solution of (P_Ω) is bounded by U_B , even in the case $a = a(t, z)$ and $\tilde{a} = \tilde{a}(t, z)$, as long as hypotheses (H1)-(H5) are satisfied. In Section 2, we review some important concepts and results. Some estimates in this section are interesting by itself. In Section 3 we get estimates assuming that $a(z) = \tilde{a}(z) = |z|^{p-2}z$ and $f(t)/t^{p-1}$ is decreasing. Indeed we prove that $\max U_B \geq \max u$ even when f is not nondecreasing. Observe that the uniqueness of solution to the problems (P_Ω) and (\tilde{P}_B) is proved in [16] for the Laplacian operator when $f(t)/t$ is decreasing. An extension of this is proved to the p -Laplacian in [13]. Hence, some results in this section can be obtained

directly from the existence of a solution associated to B that is greater than some solution associated to Ω . In Section 4, we study the behavior of solutions in the radial case. In Section 6 we obtain a bound to solutions of (1.4) and in some special case we use this comparison to show the existence of solution. In Section 7 we get some inequalities between the L^p norms of solutions of some “eigenvalue problems” and some lower bound for the distribution function of these solutions. For eigenvalue problems, the L^p estimates are established in [3], [19], and [20], where the authors obtain sharper estimates, since the constants are optimal. We are not concerned with the best constant but only with the relations between the L^p norms and the real parameter λ . We get an explicit relation for a larger class of equations and, for the typical eigenvalue problem, the estimate hold not only for the first eigenvalue of the operator but also for the others. Other authors make some similar estimates on manifolds (see e.g. [28] and [31]) for the classical eigenvalue problem, but the constant depends on the manifold and the boundary. It is also established some L^p estimates for a class of Dirichlet problems and a relation between the norms and the first eigenvalue of the domain.

2 Preliminary Results

In this section we recall some important definitions and useful results. First, if Ω is an open bounded set in \mathbb{R}^n and $u : \Omega \rightarrow \mathbb{R}$ is a measurable function, the distribution function of u is given by

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}| \quad \text{for } t \geq 0.$$

The function μ_u is non-increasing and right-continuous. The decreasing rearrangement of u , also called the *generalized inverse* of μ_u , is defined by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) \geq s\}.$$

If Ω^\sharp is the open ball in \mathbb{R}^n , centered at 0, with the same measure as Ω and ω_n is the measure of the unit ball in \mathbb{R}^n , the function

$$u^\sharp(x) = u^*(\omega_n |x|^n) \quad \text{for } x \in \Omega^\sharp$$

is the spherically symmetric decreasing rearrangement of u . It is also called the Schwarz symmetrization of u . For an exhaustive treatment of rearrangements we refer to [5], [11], [21], [30], [33], [40]. The next remark reviews important properties of rearrangements and will be necessary through this work.

Remark 2.1. *Let v, w be integrable functions in Ω and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing nonnegative function. Then*

$$\int_{\Omega} g(|v(x)|) \, dx = \int_0^{|\Omega|} g(v^*(s)) \, ds = \int_{\Omega^\sharp} g(v^\sharp(x)) \, dx.$$

Hence, if $\mu_v(t) \geq \mu_w(t)$ for all $t > t_1 > 0$, it follows that

$$\int_{t_1 < v} g(v(x)) \, dx = \int_0^{\mu_u(t_1)} g(v^*(s)) \, ds \geq \int_0^{\mu_w(t_1)} g(w^*(s)) \, ds = \int_{t_1 < w} g(w(x)) \, dx,$$

since $v^*(s) \geq w^*(s)$ for $s \leq \mu_w(t_1)$. Moreover, if $|\{v > t_2\}| \leq |\{w > t_2\}| < \infty$, $|\{v > t_1\}| = |\{w > t_1\}| < \infty$ and $|\{v > t\}| \geq |\{w > t\}|$ for all $t_1 < t < t_2$, then

$$\int_{t_1 < v \leq t_2} g(v(x)) \, dx \geq \int_{t_1 < w \leq t_2} g(w(x)) \, dx.$$

Finally, an extension of the Pólya-Szegő principle ([29], see also [15], [17], [33]) states that, if $B = B(t, z) \in C([0, \infty) \times [0, \infty))$ is increasing and convex in the variable z , then

$$\int_{\Omega} B(v(x), \nabla v(x)) \, dx \geq \int_{\Omega^\sharp} B(v^\sharp(x), \nabla v^\sharp(x)) \, dx \quad \text{for } v \geq 0 \text{ in } W_0^{1,p}(\Omega).$$

This inequality also holds if we replace Ω and Ω^\sharp by $\{t_1 < v < t_2\}$ and $\{t_1 < v^\sharp < t_2\}$, respectively. For $B(t, z) = |z|^2$, this inequality is the classical version of the Pólya-Szegő [41] principle.

Remark 2.2. For any bounded open set Ω' satisfying $|\Omega'| \leq |\Omega|$, there exists a constant $C = C(n, q, \alpha, \beta, C_*, |\Omega|, |\Omega'|)$ such that $\sup u \leq C$ for any weak solution $u \in W_0^{1,p}(\Omega')$ of $(P_{\Omega'})$. Moreover, $C = O(|\Omega'|^\rho)$ as $|\Omega'| \rightarrow 0$, where $\rho > 0$ depends only on n and p . This result is a consequence of the following two lemmas.

Lemma 2.1. Let Ω' be a bounded open set s.t. $|\Omega'| \leq |\Omega|$. If $u \in W_0^{1,p}(\Omega')$ is a nonnegative subsolution of $(P_{\Omega'})$ and conditions (H1), (H5), $C_*(|z|^q - 1) \leq \langle a(t, z), z \rangle$ for $z \in \mathbb{R}^n$, $t \in \mathbb{R}$ are satisfied, then

$$\|u\|_{L^q} \leq M(\Omega') := \left(\frac{2C_*|\Omega'|}{C_*\lambda_{B'} - \alpha} \right)^{1/q} + \left(\frac{2\beta|\Omega'|^{1/q'}}{C_*\lambda_{B'} - \alpha} \right)^{1/(q-1)},$$

where $1/q' + 1/q = 1$, B' is a ball that satisfies $|B'| = |\Omega'|$ and $\lambda_{B'}$ is the first eigenvalue of $-\Delta_q$ in B' .

Proof. Multiplying the equation by u and integrating, we get

$$\int_{\Omega'} \nabla u \cdot a(u, \nabla u) \, dx \leq \int_{\Omega'} u f(u) \, dx \leq \alpha \|u\|_q^q + \beta \|u\|_q |\Omega'|^{1/q'}.$$

Since $C_*(|z|^q - 1) \leq \langle a(t, z), z \rangle$, the first inequality of (1.1) holds. Hence

$$\|u\|_q \left[(C_*\lambda_{B'} - \alpha) \|u\|_q^{q-1} - \beta |\Omega'|^{1/q'} \right] \leq C_* |\Omega'|.$$

Studying the cases $(C_*\lambda_{B'} - \alpha) \|u\|_q^{q-1} - \beta |\Omega'|^{1/q'} \leq (C_*\lambda_{B'} - \alpha) \|u\|_q^{q-1}/2$ and $> (C_*\lambda_{B'} - \alpha) \|u\|_q^{q-1}/2$ individually, we get the result. \square

Next lemma is a particular result of Theorem 3.11 of [38] in the case $n \geq q$. For $n < q$, the estimate can be obtained following the computations of that theorem and Morrey's inequality. A sketch of the proof is done in the appendix.

Lemma 2.2. *Suppose that u satisfies the hypotheses of the preceding lemma. If $n < q$ then*

$$\sup_{\Omega'} u \leq C \|u\|_q + D |\Omega'|^{1/q},$$

where $C = C(n, q, \alpha, \beta, C_*)$ and $D = D(n, q, \alpha, \beta, C_*)$. If $n \geq q$, then

$$\sup_{\Omega'} u \leq C(|\Omega'|^{1/n} + 1)^\rho \left(\frac{\|u\|_q}{|\Omega'|^{1/q}} + |\Omega'|^{1/n} \right),$$

where $\rho = n/q$ and $C = C(n, q, \alpha, \beta, C_*)$ if $n > q$, and $\rho = \frac{\tilde{q}}{2\tilde{q}-n}$, $\tilde{q} \in (n/2, n)$, and $C = C(n, \alpha, \beta, C_*, \tilde{q})$ if $n = q$.

From these two lemmas we get, for $n < q$, that

$$\sup_{\Omega'} u \leq CM(\Omega') + D |\Omega'|^{1/q}, \quad (2.1)$$

where $C = C(n, q, \alpha, \beta, C_*)$ and $D = D(n, q, \alpha, \beta, C_*)$. For $n \geq q$, it follows that

$$\sup_{\Omega'} u \leq C(|\Omega'|^{1/n} + 1)^\rho \left(\frac{M(\Omega')}{|\Omega'|^{1/q}} + |\Omega'|^{1/n} \right), \quad (2.2)$$

where $C = C(n, q, \alpha, \beta, C_*)$ if $n > q$ and $C = C(n, \alpha, \beta, C_*, \tilde{q})$ if $n = q$. Since $\lambda_{B'} = \lambda_{B_1}/|B'|^{q/n}$, where B_1 is the unit ball, we have

$$M(\Omega') \leq E |\Omega'|^{\frac{1}{q} + \frac{1}{n}} \quad \text{if } |\Omega'| \leq |\Omega|,$$

where E is a constant that depends only on n, q, α, β, C_* and $|\Omega|$. Using this and inequalities (2.1) and (2.2), we obtain

$$\sup u \leq C |\Omega'|^{1/q} \quad \text{for } n < q \quad \text{and} \quad \sup u \leq C |\Omega'|^{1/n} \quad \text{for } n \geq q, \quad (2.3)$$

where C depends only on n, q, α, β, C_* , and Ω . Hence, if (Ω_n) is a sequence of domains such that $|\Omega_n| \rightarrow 0$ and (u_n) a sequence of solutions of (P_{Ω_n}) , then $\sup |u_n| \leq C |\Omega_n|^\sigma \rightarrow 0$, where $\sigma = 1/q$ or $\sigma = 1/n$.

Now we recall some well-known results that appear in many forms.

Lemma 2.3. *Let u be a weak solution of (P_Ω) in $W_0^{1,p}$. Then*

$$\int_{\Omega_t} -u f(u) + \nabla u \cdot a(u, \nabla u) \, dx = -t \int_{\Omega_t} f(u) \, dx \quad \forall t \geq 0,$$

where $\Omega_t = \{x \in \Omega : u(x) > t\}$.

Proof. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $\psi(s) = (s - t)\chi_{\{s > t\}}(s)$. Consider $\varphi : \Omega \rightarrow \mathbb{R}$ given by $\varphi(x) = \psi(u(x))$. Since ψ is a Lipschitz function and $t > 0$, $\varphi \in W_0^{1,p}$. Furthermore,

$$\varphi = (u - t)\chi_{\{u > t\}} \quad \text{and} \quad \nabla \varphi = \chi_{\{u > t\}} \nabla u.$$

Then, since u is a weak solution of (P_Ω) ,

$$\int_{\Omega} \chi_{\{u > t\}} \nabla u \cdot a(u, \nabla u) \, dx = \int_{\Omega} f(u)(u - t)\chi_{\{u > t\}} \, dx,$$

proving the lemma. \square

Lemma 2.4. *Assuming the same hypotheses as in the last lemma,*

$$\int_{\{u=t\}} \frac{\nabla u \cdot a(u, \nabla u)}{|\nabla u|} \, dH^{n-1} = \int_{\Omega_t} f(u) \, dx$$

for almost every $t \geq 0$. If u satisfies $u = c$ on $\partial\Omega$, $c \in \mathbb{R}$, then this identity holds for almost every $t \geq c$.

Proof. For $t_1 < t_2$, from Lemma 2.3, we get

$$\begin{aligned} \int_{A_{t_1 t_2}} -u f(u) + \nabla u \cdot a(u, \nabla u) \, dx &= t_2 \int_{\Omega_{t_2}} f(u) \, dx - t_1 \int_{\Omega_{t_1}} f(u) \, dx \\ &= (t_2 - t_1) \int_{\Omega_{t_2}} f(u) \, dx - t_1 \int_{A_{t_1 t_2}} f(u) \, dx, \end{aligned}$$

where $A_{t_1 t_2} = \{t_1 < u \leq t_2\}$. Then,

$$\int_{A_{t_1 t_2}} \nabla u \cdot a \, dx = (t_2 - t_1) \int_{\Omega_{t_2}} f(u) \, dx + \int_{A_{t_1 t_2}} (u - t_1) f(u) \, dx. \quad (2.4)$$

Hence, using the coarea formula, we obtain

$$\int_{t_1}^{t_2} \int_{\{u=t\}} \frac{(\nabla u \cdot a)|\nabla u|^{-1}}{t_2 - t_1} \, dH^{n-1} \, dt = \int_{\Omega_{t_2}} f(u) \, dx + \frac{\int_{A_{t_1 t_2}} (u - t_1) f(u) \, dx}{t_2 - t_1}.$$

Making $t_2 \rightarrow t_1$, the integral in the left hand side converges to the integrand for almost every t_1 and the integral over Ω_{t_2} converges to a integral over Ω_{t_1} . The last integral goes to zero, since

$$\left| \int_{A_{t_1 t_2}} \frac{(u - t_1)}{t_2 - t_1} f(u) \, dx \right| < \left| \int_{A_{t_1 t_2}} f(u) \, dx \right| \leq f(t_2) |A_{t_1 t_2}| \rightarrow 0,$$

completing the proof. For the case $u = c$ on $\partial\Omega$, note that $u - c \in W_0^{1,p}(\Omega)$ is a weak solution of $-\operatorname{div} \bar{a}(v, \nabla v) = \tilde{f}(v)$, where $\bar{a}(t, z) = a(t + c, z)$ and $\tilde{f}(t) = f(t + c)$. Then, from the previous case, we get result. \square

The following statement is a direct consequence of Brothers and Ziemer's result (see Lemma 2.3 and Remark 4.5 of [17]).

Proposition 2.1. *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative function and suppose that $a = a(z)$, a satisfy (H3), $a(z) \cdot z \in C^2(\mathbb{R}^n \setminus \{0\})$, $(a(z) \cdot z)^{1/p}$ is convex. If the symmetrization u^\sharp is equal to some radial solution of (P_B) on $\Omega_{t_1 t_2}^\sharp = \{x \in \Omega^\sharp : t_1 < u^\sharp(x) < t_2\}$ and*

$$\int_{t_1 < u < t_2} \nabla u \cdot a(\nabla u) dx = \int_{t_1 < u^\sharp < t_2} \nabla u^\sharp \cdot a(\nabla u^\sharp) dx,$$

for some $0 \leq t_1 < t_2 \leq \max u < +\infty$, then there is a translate of u^\sharp which is almost everywhere equal to u in $\{t_1 < u < t_2\}$. ((P_B) is the problem (\tilde{P}_B) with \tilde{a} replaced by a .)

Proof. Let U_1 be the radial solution of (P_B) such that $u^\sharp = U_1$ on $\Omega_{t_1 t_2}^\sharp$. From Lemma 2.4,

$$\int_{\partial B_t} a(\nabla U_1) \cdot n dS = \int_{B_t} f(U_1) dx > 0 \quad \text{for any } t \in [0, \max U_1),$$

where $B_t = \{x : U_1(x) > t\}$. Hence $a(\nabla U_1) \neq 0$ and, therefore, $\nabla U_1(x) = 0$ for any $x \neq 0$. Then $\nabla u^\sharp(x) \neq 0$ on the closure of $\Omega_{t_1 t_2}^\sharp$. Since $|\{\nabla U_1 = 0\}| = 0$, according to a result of Brothers and Ziemer (see Lemma 2.3 and Remark 4.5 of [17]), the equality between the Dirichlet integrals holds only if u is equal to some translation of u^\sharp almost everywhere on $\{t_1 < u < t_2\}$. \square

Next we present some comparison results about solutions.

Lemma 2.5. *Consider the radial functions $u_1(x) = w_1(|x|) \in C^1(B_{R_1})$ and $u_2(x) = w_2(|x|) \in H^1(B_{R_2})$, where B_{R_i} is the ball centered at 0 with radius R_i , $w_1 : [0, R_1] \rightarrow \mathbb{R}$ is decreasing, $w_1'(r) < 0$ for $r > 0$, $w_2 : [0, R_2] \rightarrow \mathbb{R}$ is nonincreasing, and $R_1 > R_2$. Suppose that $m = w_1(R_1) = w_2(R_2)$ and*

$$\int_{\{u_1=t\}} \frac{a(u_1, \nabla u_1) \cdot \nabla u_1}{|\nabla u_1|} dH^{n-1} \geq \int_{\{u_2=t\}} \frac{a(u_2, \nabla u_2) \cdot \nabla u_2}{|\nabla u_2|} dH^{n-1} \quad (2.5)$$

for almost all $t \in [m, +\infty)$, where H^{n-1} is the $(n-1)$ -dimensional Hausdorff measure and $a = a(t, z)$ is a function that satisfies (H3). Then $u_1 > u_2$ in $B_{R_2} \setminus \{0\}$.

Proof. We prove by contradiction. So there exists some $r_0 \in (0, R_2)$ such that $w_1(r_0) \leq w_2(r_0)$. The hypotheses imply that

$$w_1(R_2) > w_1(R_1) = w_2(R_2).$$

Hence, from the continuity of w_1 and w_2 , we can assume that

$$w_1(r_0) = w_2(r_0) \quad \text{and} \quad w_1(r) > w_2(r) \quad \text{for } r \in (r_0, R_2]. \quad (2.6)$$

Now defining $b(t, |z|) = |a(t, z)|$, we have

$$b(u_i, |\nabla u_i|) = |a(u_i, \nabla u_i)| = \frac{a(u_i, \nabla u_i) \cdot \nabla u_i}{|\nabla u_i|} \quad \text{for } i = 1, 2.$$

Observe that $b = b(t, s) \in C^0(\mathbb{R} \times [0, +\infty)) \cap C^1(\mathbb{R} \times (0, +\infty))$ is positive for $s \neq 0$ and increasing in s . Hence, using (2.5) and $w'_i(|x|) = -|\nabla u_i(x)|$, we get

$$b(t, -w'_1(r_1(t))) r_1^{n-1}(t) \geq b(t, -w'_2(r_2(t))) r_2^{n-1}(t)$$

a.e. on $I = [m, t_0]$, where $t_0 = w_1(r_0) = w_2(r_0)$ and r_i is some kind of inverse of w_i given by $r_i(t) = \inf\{r \mid w_i(r) \leq t\} = (\mu_{u_i}(t)/\omega_n)^{1/n}$. Notice that r_1 is decreasing and r_2 is nonincreasing and, therefore, they are differentiable a.e. on I with $r'_i(t) = (w'_i(r_i(t)))^{-1}$. Then

$$b\left(t, -\frac{1}{r'_1(t)}\right) r_1^{n-1}(t) \geq b\left(t, -\frac{1}{r'_2(t)}\right) r_2^{n-1}(t) \quad \text{a.e. on } I.$$

Defining $d : \mathbb{R} \times (-\infty, 0) \rightarrow \mathbb{R}$ by $d(t, y) = [b(t, -1/y)]^{1/(n-1)}$, we obtain

$$d(t, r'_1(t)) r_1(t) \geq d(t, r'_2(t)) r_2(t) \quad \text{a.e. on } I$$

and, therefore,

$$d(t, r'_1(t)) (r_1 - r_2) \geq (d(t, r'_2(t)) - d(t, r'_1(t))) r_2 \quad \text{a.e. on } I.$$

Since $r_2 \geq r_0 > 0$, $d(t, r'_1(t))$ is continuous and positive in I , and $r_1 - r_2 \geq 0$, there exist $c_1 > 0$ such that

$$c_1 (r_1 - r_2) \geq (d(t, r'_2(t)) - d(t, r'_1(t))) r_0 \quad \text{a.e. on } I. \quad (2.7)$$

We prove now that, for some suitable constant $C > 0$,

$$C(r_1 - r_2) \geq r'_2 - r'_1 \quad \text{a.e. on } I. \quad (2.8)$$

For that note first that if $t \in I$ satisfies $r'_2(t) \leq r'_1(t)$, the inequality is trivial for any $C > 0$ since $r_1 \geq r_2$ on I . In the case $r'_2(t) > r'_1(t)$,

$$d(t, r'_2) - d(t, r'_1) = \int_{r'_1}^{r'_2} \frac{\partial d}{\partial y}(t, y) dy \geq \int_{r'_1}^{\frac{r'_2 + r'_1}{2}} \frac{[b(t, -\frac{1}{y})]^{\frac{2-n}{n-1}}}{n-1} \cdot \frac{b_s(t, -\frac{1}{y})}{y^2} dy$$

since the integrand is positive and $(r'_2 + r'_1)/2 \leq r'_2$. From the C^1 regularity of w_1 and $w'_1 < 0$, it follows that the interval $[r'_1(t), r'_1(t)/2]$ is contained in some interval $[y_1, y_2]$, where $y_2 < 0$, for any $t \in I$. Then

$$[r'_1, (r'_1 + r'_2)/2] \subset [r'_1, r'_1/2] \subset [y_1, y_2] \subset (-\infty, 0) \quad \text{for any } t \in I,$$

and, using that $|a|$ and $\partial_s |a(t, sz)|$ are positive and continuous for $s, z \neq 0$, we get

$$b\left(t, -\frac{1}{y}\right) \geq \min_{[y_1, y_2]} b\left(t, -\frac{1}{y}\right) \geq E_1 := \min_{\frac{1}{|y_1|} \leq |z| \leq \frac{1}{|y_2|}} |a(t, z)| > 0$$

and

$$b_s \left(t, -\frac{1}{y} \right) \geq \min_{[y_1, y_2]} b_s \left(t, -\frac{1}{y} \right) \geq E_2 := \min_{\frac{1}{|y_1|} \leq |z| \leq \frac{1}{|y_2|}} \partial_s |a(t, sz)| \Big|_{s=1} > 0$$

for $y \in [r'_1, (r'_1 + r'_2)/2]$. Hence,

$$d(t, r'_2) - d(t, r'_1) \geq \int_{r'_1}^{(r'_1 + r'_2)/2} \frac{E_1^{\frac{2-n}{n-1}}}{n-1} \cdot \frac{E_2}{y^2} dy \geq \frac{E_1^{\frac{2-n}{n-1}} E_2}{(n-1) y_1^2} \cdot \frac{(r'_2 - r'_1)}{2}.$$

From this and (2.7), we get (2.8) with $C = 2c_1(n-1)y_1^2/(r_0 E_1^{\frac{2-n}{n-1}} E_2)$. Multiplying (2.8) by e^{Ct} , it follows that

$$\frac{d}{dt}(r_1 e^{Ct}) \geq \frac{d}{dt}(r_2 e^{Ct}) \quad \text{a.e. on } I.$$

Observe that $\int_m^{t_0} (r_2 e^{Ct})' dt \geq r_2 e^{Ct} \Big|_m^{t_0}$, since r_2 is decreasing and e^{Ct} is a C^1 function. To prove that, we can split $r_2 e^{Ct}$ into a singular function and an absolutely continuous function, apply the Fundamental Theorem of Calculus, obtaining an identity for the second part and, using a sequence of increasing C^1 functions that converges uniformly to r_2 , an inequality for the first part. Therefore

$$r_1 e^{Ct} \Big|_m^{t_0} = \int_m^{t_0} \frac{d}{dt}(r_1 e^{Ct}) dt \geq \int_m^{t_0} \frac{d}{dt}(r_2 e^{Ct}) dt \geq r_2 e^{Ct} \Big|_m^{t_0}.$$

Hence, using $r_1(t_0) = r_2(t_0) = r_0$, we get $r_1(m) \leq r_2(m)$. But this contradicts $r_1(m) = R_1 > R_2 = r_2(m)$. \square

3 Comparison results to the p-laplacian

We treat in this section the special case where the differential part of (P_Ω) and (\tilde{P}_B) is the p -laplacian operator and, in addition to the hypotheses (H1) and (H5), we suppose that $f(t)/t^{p-1}$ is decreasing. Then, we can obtain a solution to the problem (\tilde{P}_B) minimizing the functional

$$J_B(v) = \int_B \frac{1}{p} |\nabla v|^p - F(v) dx, \quad (3.1)$$

where $F(t) = \int_0^t f(s) ds$. Let \tilde{U}_B be a minimum of J_B . Since $f(t)/t^{p-1}$ is decreasing, \tilde{U}_B is the unique solution to (\tilde{P}_B) (see [16] and [13]). Then $\tilde{U}_B = U_B$, where U_B is defined in (1.2). This uniqueness result is applied only in Theorem 3.2.

Remark 3.1. For any ball $B_r \subset B$ and $\alpha \in \mathbb{R}$, there is a radial minimizer w of the functional

$$J_{B_r}(v) = \int_{B_r} \frac{1}{p} |\nabla v|^p - F(v) dx,$$

such that $w \equiv \alpha$ on ∂B_r . Moreover, if u and w are minimizers of J_{B_r} and $u > w$ on ∂B_r , then $u > w$ in B_r and $J_{B_r}(u) < J_{B_r}(w)$.

The first part of this remark follows from classical arguments of compactness. To prove that $u > w$, observe that for any open subset $A \subset B_r$, u and w minimizes the corresponding functional J_A in the set of functions with prescribed boundary data $v = u$ and $v = w$ on ∂A , respectively. Hence, if $u < w$ for some open set, then the $v_0 = \max\{u, w\}$ is also a minimizer of J_{B_r} in $W^{1,p}(B_r)$ and u touches v_0 by below in B_r , contradicting $-\Delta_p v_0 = f(v_0) \geq f(u) = -\Delta_p u$ and the maximum principle. The inequality $J_{B_r}(u) < J_{B_r}(w)$ is a consequence of $J_{B_r}(v) < J_{B_r}(w)$ for any $v \in W^{1,p}(B_r)$ since F is increasing. The next result does not requires that f is nondecreasing.

Theorem 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, B be a ball such that $|B| = |\Omega|$, and u be a weak solution of (P_Ω) , where $\operatorname{div}(a(\nabla u)) = \Delta_p u$ and f is a nonnegative locally Lipschitz function, possibly non-monotone, such that $f(t)/t^{p-1}$ is decreasing on $(0, +\infty)$. Then,

$$\max u \leq \max U_B,$$

where U_B is the minimizer of the functional given by (3.1).

Proof. Let u^\sharp be the Schwarz symmetrization of u . Defining $\Omega_t^\sharp = \{u^\sharp > t\}$, we have that $|\Omega_t^\sharp| = |\Omega_t|$. Therefore, Remark 2.1 implies that

$$\int_{\Omega_t} F(u) dx = \int_{\Omega_t^\sharp} F(u^\sharp) dx \quad \text{for } t \geq 0.$$

We also know that

$$\int_{\Omega_t} |\nabla u|^p dx \geq \int_{\Omega_t^\sharp} |\nabla u^\sharp|^p dx. \quad (3.2)$$

Then,

$$\int_{\Omega_t} \frac{|\nabla u|^p}{p} - F(u) dx \geq \int_{\Omega_t^\sharp} \frac{|\nabla u^\sharp|^p}{p} - F(u^\sharp) dx. \quad (3.3)$$

Now suppose that for some $t \geq 0$, we have $|\Omega_t| = |B_t|$, where $B_t = \{U_B > t\}$. In this case $B_t = \Omega_t^\sharp$ and

$$\int_{\Omega_t^\sharp} \frac{|\nabla u^\sharp|^p}{p} - F(u^\sharp) dx \geq \int_{B_t} \frac{|\nabla U_B|^p}{p} - F(U_B) dx, \quad (3.4)$$

otherwise the function $\tilde{u} : B \rightarrow \mathbb{R}$ given by $\tilde{u} = u^\sharp \chi_{B_t} + U_B \chi_{B_t^c}$ is the minimum of J_B . Then, from (3.3) and (3.4), it follows that

$$\int_{\Omega_t} \frac{|\nabla u|^p}{p} - F(u) dx \geq \int_{B_t} \frac{|\nabla U_B|^p}{p} - F(U_B) dx.$$

Hence, using Lemma 2.3 and the fact that u and U_B are solutions, we get

$$\int_{\Omega_t} \frac{uf(u) - tf(u)}{p} - F(u) \, dx \geq \int_{B_t} \frac{U_B f(U_B) - tf(U_B)}{p} - F(U_B) \, dx. \quad (3.5)$$

Define $h_t : [t, +\infty) \rightarrow \mathbb{R}$ by

$$h_t(s) = \frac{(s-t)f(s)}{p} - F(s). \quad (3.6)$$

Note that $h_t(s)$ is decreasing for $s \geq t$, since

$$h'_t(s) = \frac{(s-t)f'(s)}{p} - \frac{(p-1)f(s)}{p} = \frac{(s-t)^p}{p} \left(\frac{f(s)}{(s-t)^{p-1}} \right)' < 0.$$

Furthermore, as $h_t(t) \leq 0$, $h_t(s) < 0$ for $s > t$. Therefore, from (3.5), we have

$$\int_{\Omega_t} h_t(u) \, dx \geq \int_{B_t} h_t(U_B) \, dx, \quad (3.7)$$

where h_t is decreasing and negative. Suppose that $\max u > \max U_B$. Since $|\Omega| = |B|$, the function $\mu_B(t) = |\{U_B > t\}|$ is continuous and $\mu_u(t)$ is right continuous, there is $t_0 \geq 0$ such that $\mu_u(t_0) = \mu_B(t_0)$ and $\mu_u(t) > \mu_B(t)$ for $t > t_0$. Then,

$$|\{-h_{t_0} \circ u > s\}| > |\{-h_{t_0} \circ U_B > s\}| \quad \text{for } s > -h_{t_0}(t_0),$$

since $-h_{t_0}$ is an increasing function. Thus, by Fubini's Theorem,

$$-\int_{\Omega_{t_0}} h_{t_0}(u) \, dx > -\int_{B_{t_0}} h_{t_0}(U_B) \, dx,$$

contradicting (3.7). \square

Remark 3.2. This result can be extended to the problem (1.5) observing first that $q(t) := (f(t) - k(t))/t$ is decreasing. If $q(t) > 0$ for any $t > 0$, it is immediate from the theorem that $\max u \leq \max U$, where u solves (1.5) and $U \in W_0^{1,p}(B)$ is the solution of the symmetrized problem $-\Delta_p V + k(V) = f(V)$ in B . If $q(t_0) = 0$ for some $t_0 \geq 0$, the maximum principle implies that $u, U \leq t_0$. Hence taking u_m and U_m , the sequence of solutions of $-\Delta_p v = \max\{f(v) - k(v), 0\} + 1/m$ in Ω and B respectively, we have $u_m \leq U_m$, $u_m \rightarrow u$ and $U_m \rightarrow U$ monotonically, proving the inequality. A related result with this one is stated in [23]. For instance, if f is a positive constant, Theorem 2 of that work gives more relations between u and U .

Corollary 3.1. Assuming the same hypotheses as in Proposition 3.1, if Ω is not a ball, then

$$\max u < \max U_B.$$

Proof. If Ω is not a ball, Proposition 2.1 implies that inequalities (3.2) and (3.7) are strict for $t = 0$. Therefore, there is $t > 0$ such that

$$|\Omega_t| < |B_t|.$$

Note that the function $v = u - t$ satisfies

$$\begin{cases} -\Delta_p v &= \tilde{f}(v) & \text{in } \Omega_t \\ v &= 0 & \text{on } \partial\Omega_t, \end{cases} \quad (3.8)$$

where \tilde{f} is given by $\tilde{f}(s) = f(s + t)$. If B' and B are concentric balls and $|B'| = |\Omega_t|$, then $|B'| < |B_t|$ and $B' \subset B_t$. Since $U_B = t$ on ∂B_t , we get from the maximum principle that $U_B > t$ on $\partial B'$. Hence, using Remark 3.1, there is a function $w : B' \rightarrow \mathbb{R}$ that minimizes $J_{B'}$ under the condition $w \equiv t$ on $\partial B'$ and, therefore, the function $V_{B'} = w - t$ is the solution of (3.8) with Ω_t replaced by B' . Furthermore, $w < U_B$. Since \tilde{f} satisfies all hypotheses required in Theorem 3.1,

$$\max v \leq \max V_{B'}.$$

Hence,

$$\max_{\Omega} u = \max_{\Omega_t} (v + t) \leq \max_{\Omega_t} (V_{B'} + t) = \max_{\Omega_t} w < \max_{\Omega} U_B$$

proving the result. \square

Remark 3.3. Suppose that $u \in W_0^{1,p}(\Omega)$ is a solution of

$$-\operatorname{div}(MDu|Du|^{p-2}) = f(u), \quad (3.9)$$

where $M(x) = (a_{ij}(x))$ is a matrix with measurable bounded entries such that, $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2$. Observing that

$$\tilde{J}(v) := \int_{\Omega} \langle MDv, Dv \rangle \frac{1}{p} |\nabla v|^{p-2} - F(v) \, dx \geq \int_{\Omega} \frac{1}{p} |\nabla v|^p - F(v) \, dx,$$

and repeating the arguments of Theorem 3.1, we get $\max u \leq \max U_B$. Notice that M can be nonsymmetric.

Next result is some sort of maximum principle for the distribution function. The proof will be given for a more general case in Section 5, Proposition 5.2.

Proposition 3.1. Suppose that $u \in W_0^{1,p}(\Omega)$ and $U \in W_0^{1,p}(B)$ satisfy $-\Delta_p u = f(u)$ and $-\Delta_p U = f(U)$, where f is a nondecreasing locally Lipschitz function, positive on $(0, +\infty)$. If $u^\sharp \leq U$ and $u^\sharp \not\equiv U$, then $u^\sharp < U$ on B .

Next theorem, in the case $p = 2$ and $f(0) > 0$, is a consequence of a result, which establishes that the symmetrization of the minimal solution associated to Ω is smaller or equal than the one associated to the corresponding ball (see [11], [37]), and the uniqueness of solution when $f(t)/t$ is decreasing (see [16]). For

general p , we can apply a similar argument to compare the minimal solutions (see [32]) and the uniqueness result obtained for the case that $f(t)/t^{p-1}$ is decreasing (see [13]).

Also it can be proved in a independent way using the main result of Section 5 and the uniqueness of solution to this problem.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, B be a ball such that $|B| = |\Omega|$, and u be a weak solution of (P_Ω) , where $\operatorname{div}(a(\nabla u)) = \Delta_p u$ and f is a nonnegative increasing locally Lipschitz function, such that $f(t)/t^{p-1}$ is decreasing on $(0, +\infty)$. Then,*

$$|\{u > t\}| < |\{U_B > t\}| \quad \forall t \in (0, \max U_B],$$

unless Ω is a ball.

4 Study of the radial solutions

We study now a Dirichlet problem, where the domain is a ball, and we need some additional hypothesis:

(H6) there is some $\mu \in [0, 2)$ such that $\frac{d}{ds}|a(t, sw)| \geq |a(t, sw)|^\mu$ for $s > 0$ small and w unit vector of \mathbb{R}^n .

The following theorem is the main result of this section.

Theorem 4.1. *Let $B' = B_{R_0}$ be a open ball in \mathbb{R}^n satisfying $|B'| \leq |\Omega|$ and suppose that \tilde{a} and f satisfy conditions (H1)-(H6). If $f(0) > 0$ and $m \geq 0$, then there exists a solution $U_{B'}$ to the problem $(\tilde{P}_{B'})$ with $U_{B'} = m$ on $\partial B'$ such that, for any radial solution U of $(\tilde{P}_{B''})$ with $0 \leq U \leq m$ on $\partial B''$,*

$$U_{B'} > U \quad \text{in } B'',$$

where $B'' \subsetneq B'$ are concentric open balls. The same holds in the case $B'' = B'$ if U and $U_{B'}$ are different.

Remark 4.1. *Suppose that the hypotheses of this theorem holds and U is a radial weak solution of $(\tilde{P}_{B''})$. We will see that U is a classical solution in $B'' \setminus \{0\}$. First using the ACL characterization of Sobolev functions (see e.g. [47]) and a local diffeomorphism between the Cartesian and the polar system of coordinates, it follows that U is absolutely continuous on closed radial segments that does not contain the origin. Hence the set $\{U < t\}$ is open in B' for any $t \in \mathbb{R}$. Indeed, these sets are rings of the form $\{x \in B' : r_t < |x| < R_0\}$, otherwise there is a ring $\mathcal{R} = \{r_1 < |x| < r_2\}$ contained in $\{U < t\}$, such that $U = t$ on $\partial \mathcal{R}$, for which the test function $\varphi(x) = (t - U(x))\chi_{\mathcal{R}}(x) \in W_0^{1,p}$ satisfies*

$$0 \geq - \int_{\mathcal{R}} \nabla U \cdot \tilde{a}(U, \nabla U) dx = \int_{\mathcal{R}} \nabla \varphi \cdot \tilde{a}(U, \nabla U) dx = \int_{\mathcal{R}} f(U) \varphi dx > 0,$$

that is a contradiction. Hence, U is a nonincreasing radially symmetric function. Observe also that if U is constant in some ring, then taking a nonnegative function with a compact support in this ring, we get a contradiction as before. Then U is strictly decreasing in the radial direction. This conclusion can be obtained more easily for operators where the maximum principle holds.

Notice now that for a given ring $\mathcal{R} = \{r_1 < |x| < r_2\}$, taking the radial test function $\varphi_{R,h}(|x|) = \chi_{[0,R-h]}(|x|) + \left(\frac{R+h}{2h} - \frac{|x|}{2h}\right) \chi_{(R-h,R+h]}(|x|)$, for $h > 0$ and $R \in (r_1, r_2)$, we get

$$n\omega_n \int_{R-h}^{R+h} \frac{b(U, -\partial_r U)}{2h} r^{n-1} dr = \int_{B'} \nabla \varphi_{R,h} \cdot \tilde{a}(U, \nabla U) dx = \int_{B'} f(U) \varphi_{R,h} dx,$$

where $b(t, |z|) = |\tilde{a}(t, z)|$ and ω_n is the volume of the unit ball. Making $h \rightarrow 0$, from the Lebesgue Differentiation Theorem, it follows that

$$n\omega_n b(U(R), -\partial_r U(R)) R^{n-1} = \int_{B_R} f(U) dx \geq \int_{B_{r_1}} f(0) dx > 0 \quad (4.1)$$

for almost every $R \in (r_1, r_2)$ and then, using (H3), we get that $|\nabla U| \geq c$ a.e. in \mathcal{R} , where c is some positive constant that depends on \mathcal{R} . Thus U is a solution of a uniformly elliptic equation in this ring and, therefore, a $C^{2,\alpha}$ function in \mathcal{R} for any $\alpha \in (0, 1)$. Moreover, from (2.3), U is bounded and, from its monotonicity in the radial direction, it can be defined continuously on 0. In fact, using (4.1), we can prove that U is differentiable at the origin and its derivative is zero.

Due to this regularity of U and (H3), we have $\tilde{a}(U, \nabla U) = \tilde{e}(U, |\nabla U|) \nabla U$ for some function $\tilde{e} : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ and $U(x) = w(|x|)$ for some function $w : [0, R_1] \rightarrow \mathbb{R}$ that satisfies, in the classical sense,

$$\begin{cases} (\tilde{e} w')' + \frac{n-1}{r} \tilde{e} w' &= -f(w) \quad \text{for } r \in [0, R_1] \\ w'(0) &= 0 \\ w(R_1) &= 0, \end{cases} \quad (4.2)$$

where R_1 is the radius of B'' and $'$ denotes d/dr . To prove the existence of solution to this problem, we consider the following one:

$$\begin{cases} (\tilde{e}(w(r), |w'(r)|) w'(r))' + \frac{n-1}{r} \tilde{e} w'(r) &= -f(w(r)) \\ w'(0) &= 0 \\ w(0) &= h, \end{cases} \quad (4.3)$$

where $h > 0$ is given. If \tilde{e} depends only on z , according to Proposition A1 of [24], there exists $\delta > 0$ and a positive local solution $w_h : [0, \delta) \rightarrow \mathbb{R}$ to (4.3). In the general case, consider first the problem (4.3) with \tilde{e} replaced by $e_0(|z|) = \tilde{e}(h, |z|)$, that has a local solution w_0 defined on $[0, \delta_0)$ as in the previous case. Then, for $k \in \mathbb{N}$, take $\delta_k \leq \delta_0$ such that $w_0(r) \geq h - h/k$ on $[0, \delta_k]$ and define e_k such that $\tilde{a}_k(t, z) := e_k(t, |z|)z$ satisfies (H3), (H4), (H6) and

$$e_k(t, |z|) = \begin{cases} e_0(|z|) = \tilde{e}(h, |z|) & \text{for } t \in [h - h/k, +\infty) \\ \tilde{e}(t, |z|) & \text{for } t \in (-\infty, h - 2h/k]. \end{cases}$$

Hence w_0 is a solution to (4.3) on $[0, \delta_k]$ with \tilde{e} replaced by e_k and, from (4.1), w_0 is decreasing and $\frac{dw_0}{dr}(\delta_k) \neq 0$. Since (H3) implies that $s \rightarrow |\tilde{a}_k(t, sw)|$ is increasing for any w , the classical ODE theory implies that we can extend w_0 for a larger interval. Indeed, while some extension is positive, it can be continued to a bigger interval. Since $f(0) > 0$ and \tilde{a}_k satisfies (H4), integrating $(r^{n-1}e_k(w(r), |w'(r)|)w'(r))' = -r^{n-1}f(w(r))$, we conclude that for any positive continuation $\bar{w}_k : [0, \delta) \rightarrow \mathbb{R}$ of w_0 , the right end point satisfies

$$\bar{\delta} \leq C := \frac{nC^*}{f(0)} \left[1 + \left(\frac{p}{p-1} \cdot \frac{hf(0)}{nC^*} \right)^{\frac{p-1}{p}} \right]. \quad (4.4)$$

Hence, there exists a continuation $w_k : [0, R_k] \rightarrow \mathbb{R}$ such that $w_k(R_k) = 0$ and is positive on $[0, R_k)$. Observe now that, using the same idea as in the estimate (4.1), we get that $|w'_k|$ is uniformly bounded by above. Hence some subsequence converge uniformly for some nondecreasing function $w_h : [0, R_h] \rightarrow \mathbb{R}$ that is positive in $[0, R_h)$ and vanishes at R_h . Indeed, applying again a similar computation as in (4.1) and using the positivity of $|\partial_s \tilde{a}_k(t, sz)|$ for $s, t \neq 0$ from (H3), it follows that w'_k are equicontinuous in compact sets of $[0, R_h)$ for k large. (More precisely, the Lipschitz norm of w'_k are uniformly bounded in compact sets of $(0, R_h)$ and $w'_k(r)$ are uniformly close to 0 for r small.) Hence, some subsequence converge uniformly for w_h in the C^1 norm for compact sets of $[0, R_h)$. Hence, due to the regularity of \tilde{a} and the definition of \tilde{a}_k , $U_h(x) := w_h(|x|)$ is the weak solution of $-\operatorname{div} \tilde{a}(v, \nabla v) = f(v)$ in B_{R_h} . Then, as we observed previously, U_h is a classical solution, and satisfies $U_h(0) = h$ since $w_k(0) = h$. Moreover, following the same argument of Proposition A4 of [24] for \tilde{a} that depends also on t , for each $h > 0$, such solution U_h and radius R_h are unique. Let us represent this correspondence by $\Psi = (\Psi_1, \Psi_2)$, where $\Psi_1(h) = R_h$ and $\Psi_2(h) = U_h$.

Observe that $R_h \leq C$, where C is given by (4.4). Using this, the equicontinuity of the first derivative of solutions, Arzelà-Ascoli Theorem and uniqueness for (4.3), we get the following result.

Lemma 4.1. *The function Ψ_1 is continuous on $(0, +\infty)$. Furthermore, for any $h_0 > 0$, $\varepsilon > 0$ and K compact subset of $B_{R_{h_0}}$, there exists $\delta > 0$ such that $\|\Psi_2(h) - \Psi_2(h_0)\|_{C^1(K)} \leq \varepsilon$ if $|h - h_0| < \delta$.*

We can also improve estimate (4.4) in the following sense.

Lemma 4.2. *Given $M > 0$, there exists some continuous increasing function $\Theta_M : [0, M] \rightarrow \mathbb{R}$ s.t. $\Theta_M(0) = 0$ and $R_h \leq \Theta_M(h)$ for $h \leq M$, where $R_h = \Psi_1(h)$, i.e., R_h is the point s.t. the nonnegative solution w of (4.3) vanishes.*

Proof. Integrating $(r^{n-1}e(w(r), |w'(r)|)w'(r))' = -r^{n-1}f(w(r))$ from 0 to $R \leq R_h$, we get

$$|\tilde{a}(w(R), |w'(R)|z)| = -e(w(R), |w'(R)|)w'(R) \geq \frac{f(0)R}{n}$$

for any $|z| = 1$. Since $s \mapsto |\tilde{a}(t, sz)|$ is continuous, strictly increasing in $[0, +\infty)$ and vanishes at $s = 0$, where $t \in [0, M]$, the function $\rho(s) := \sup_{t \in [0, M]} |\tilde{a}(t, sz)|$ also satisfies these hypotheses. Using that $w(R) \leq h \leq M$,

$$\rho(-w'(R)) \geq \frac{f(0)R}{n}.$$

Taking the inverse of ρ and integrating from 0 to R_h ,

$$h = w(0) - w(R_h) = \int_0^{R_h} -w'(R) dR \geq \int_0^{R_h} \rho^{-1} \left(\frac{f(0)R}{n} \right) dR.$$

Observe that

$$R_h \mapsto \int_0^{R_h} \rho^{-1} \left(\frac{f(0)R}{n} \right) dR$$

is invertible, since is increasing, positive and vanishes at 0. Hence, we get the result defining Θ_M as the inverse of this application. \square

Lemma 4.3. *Assuming the same hypotheses as in Theorem 4.1, there exists a solution $U_{B'}$ to the problem $(P_{B'})$ with $U_{B'} = 0$ on $\partial B'$, such that*

$$\max U_{B'} \geq \max U,$$

for any radial solution U of $(P_{B''})$ satisfying $U = 0$ on $\partial B''$, where $B'' \subset B' = B_{R_0}$ are concentric balls. As a matter of fact, $U_{B'} = \Psi_2(h_0)$, where $h_0 = \max\{h \mid \Psi_1(h) = R_0\}$. Furthermore, the inequality is strict if $U \neq U_{B'}$.

Proof. First we note that Lemma 4.2 implies that

$$\Psi_1(h_1) = R_{h_1} \leq \Theta_1(h_1) < R_0 \quad \text{for small } h_1,$$

since $\Theta_1(h) \rightarrow 0$ as $h \rightarrow 0$. We can also prove that $\Psi_1(h_2) > R_0$ for a large h_2 . Indeed, from (2.3), any solution of $(P_{B''})$ is bounded by $C|B'|^{1/q}$ if $n < q$ or by $C|B'|^{1/n}$ if $n \geq q$. Hence,

$$\Psi_1(h) > R_0 \quad \text{for } h > M = \max\{C|B'|^{1/q}, C|B'|^{1/n}\}, \quad (4.5)$$

otherwise a ball of radius $\Psi_1(h) \leq R_0$ posses a solution of height $h > M$ contradicting (2.3).

Thus, from the continuity of Ψ_1 , the set $A = \{h \mid \Psi_1(h) = R_0\}$ is not empty and is bounded by M . Then, we can define $h_0 = \max A$ and $U_{B'} = \Psi_2(h_0)$. Let U be a radial solution of $(P_{B''})$ satisfying $U = 0$ on $\partial B''$, where $B'' = B_{\tilde{R}}$ with $\tilde{R} \leq R_0$. Note that $\tilde{R} = \Psi_1(U(0))$ and, thus, inequality (4.5) implies that $U(0) \leq M$. To prove the lemma we have to show that $U(0) \leq h_0$. Suppose that $U(0) > h_0$. For $h = M + 1$, we have $\Psi_1(h) > R_0$ from (4.5). Summarizing,

$$\Psi_1(U(0)) = \tilde{R} \leq R_0 < \Psi_1(h) \quad \text{and} \quad U(0) < h.$$

Therefore, from the continuity of Ψ_1 , there exists $h_1 \in [U(0), h]$ such that $\Psi_1(h_1) = R_0$. But this contradicts $h_1 \geq U(0) > h_0$ and the definition of h_0 . Hence $U(0) \leq h_0$. Furthermore, the equality happens only if $U = U_{B'}$, since the solution of (4.3) is unique. \square

Proof. of Theorem 4.1

Possibility 1: $m = 0$

Let $U_{B'}$ be the function defined in the previous lemma and U a solution of $(P_{B''})$ with $U = 0$ on $\partial B''$, where $B'' \subset B'$ are concentric balls. The set

$$C = \{h > 0 \mid w_h := \Psi_2(h) \geq U_{B'} \text{ in } B' \text{ and } w_h \geq U \text{ in } B''\}$$

is not empty. To prove that, let $h > \max U_{B'}$ such that $h \notin C$. For instance, suppose that w_h does not satisfy $w_h \geq U_{B'}$ in B' . Using that w_h and $U_{B'}$ are continuous radial functions and $w_h(0) = h > U_{B'}(0)$, we conclude that there exists $B'' \subset B'$ such that $w_h > U_{B'}$ in B'' and $w_h = U_{B'}$ in $\partial B''$. Denoting $t_0 = U_{B'}^{-1}(B'')$, we have $t_0 \leq \max U_{B'} \leq M$, where M is given by (4.5). Hence, the function $\tilde{f}(t) = f(t + t_0)$ satisfies

$$\tilde{f}(t) \leq f(t + M) \leq \alpha(t + M)^{q-1} + \beta \leq \alpha' t^{q-1} + \beta',$$

where α' is any real in $(\alpha, C_* \lambda_B)$ and β' is a constant that depends on α' , β and M . Note that $v = w_h - t_0$ satisfies

$$-\operatorname{div}(\bar{a}(v, \nabla v)) = \tilde{f}(v),$$

where $\bar{a}(t, z) = \tilde{a}(t + t_0, z)$, with the boundary data $v = 0$ on B'' . Since \bar{a} and \tilde{f} satisfy (H1)-(H6), it follows from (2.3) that $\sup v \leq \tilde{M}$, where \tilde{M} is a constant that depends on $n, q, \alpha', \beta', C_*$, and $|\Omega|$. Thus $w_h \leq \tilde{M} + M$. This inequality also holds, by the same argument, when condition $w_h \geq U$ in B' is not satisfied. Therefore, $h \in C$ for $h > \tilde{M} + M$, proving that C is not empty.

Let $\alpha_1 = \inf C$. From the continuity of Ψ_1 and the C^1 estimate of Lemma 4.1, $R_1 = \Psi_1(\alpha_1) \geq R_0$, $w_{\alpha_1} = \Psi_2(\alpha_1) \geq U_{B'}$ in B' , and $w_{\alpha_1} \geq U$ in B'' . Hence $\alpha_1 = w_{\alpha_1}(0) \geq U_{B'}(0)$. If $\alpha_0 := U_{B'}(0) = \alpha_1$, then $w_{\alpha_1} = U_{B'}$ and, therefore, $U_{B'} \geq U$ proving the theorem. Suppose that $\alpha_1 > \alpha_0$. Then $R_1 > R_0$, otherwise $R_0 = R_1 = \Psi_1(\alpha_1)$ contradicting $\alpha_1 > \alpha_0 = \max\{\alpha \mid \Psi_1(\alpha) = R_0\}$. Let

$$d_1 = \inf_{x \in B'} (w_{\alpha_1}(x) - U_{B'}(x)) \geq 0 \quad \text{and} \quad d_2 = \inf_{x \in B''} (w_{\alpha_1}(x) - U(x)) \geq 0.$$

If $d_1 = 0$, consider $x_1 \in \bar{B}' \setminus \{0\}$ such that $w_{\alpha_1}(x_1) = U_{B'}(x_1)$. Since $R_1 > R_0$, we have $w_{\alpha_1} > 0$ in $\partial B'$ and, from $U_{B'} = 0$ in $\partial B'$, it follows that $x_1 \in B' \setminus \{0\}$. Observe also that $\nabla w_{\alpha_1}(x_1) = \nabla U_{B'}(x_1)$, since $w_{\alpha_1} \geq U_{B'}$. Then, using that w_{α_1} and $U_{B'}$ are radial, we infer from the uniqueness of solution for ODE that $w_{\alpha_1} = U_{B'}$, contradicting $w_{\alpha_1}(0) = \alpha_1 > \alpha_0 = U_{B'}(0)$. Hence $d_1 > 0$ and, by the same argument, $d_2 > 0$. These contradict Lemma 4.1 and the definition of α_1 , proving that $U_{B'} \geq U$.

Possibility 2: $m > 0$

Consider the equation

$$-\operatorname{div} \bar{a}(V, \nabla V) = \tilde{f}(V),$$

where $\bar{a}(t, z) = \tilde{a}(t + m, z)$ and $\tilde{f}(t) = f(t + m)$. Notice that \bar{a} and \tilde{f} satisfy (H1)-(H6) with the constants $n, p, q, q_0, \alpha', \beta', C_*, C^*, C_s$ and $|\Omega|$, where α' and

β' can be chosen, as in Possibility 1, s.t. $\alpha' \in (\alpha, C_* \lambda_B)$ and $\beta' = \beta'(\alpha', \beta, m)$. Then, from Possibility 1, let $\tilde{U} \in W_0^{1,p}(B')$ be the maximal solution associated to this equation. If U is a solution of $(\tilde{P}_{B''})$ with $U \leq m$ on $\partial B''$, then $U - m \leq 0$ or $U - m$ is also a solution of this equation in some ball contained in B'' . In both situations, since \tilde{U} is maximal, $\tilde{U} \geq U - m$. So we conclude Possibility 2, taking $U_{B'} = \tilde{U} + m$.

To prove the strict inequality in case $U \neq U_{B'}$, we must observe that if $U(x_0) = U_{B'}(x_0)$ at some $x_0 \in B''$, then $\nabla U(x_0) = \nabla U_{B'}(x_0)$ since $U \leq U_{B'}$. This contradicts the classical results of uniqueness of solution for ODE if $x_0 \neq 0$ and the uniqueness established by Proposition A4 of [24] if $x_0 = 0$, as we already pointed out. \square

Theorem 4.2. *Let $B' = B_{R_0}$ be a open ball in \mathbb{R}^n satisfying $|B'| \leq |\Omega|$ and suppose that \tilde{a} and f satisfy conditions (H1)-(H6). If $f(0) = 0$ and $m \geq 0$, then there exists a nonnegative solution $U_{B'}$ of $(\tilde{P}_{B'})$ with $U_{B'} = m$ on $\partial B''$, possibly null, s.t. for any radial solution U of $(\tilde{P}_{B''})$ with $U \leq m$ on $\partial B''$,*

$$U_{B'} \geq U \quad \text{in } B'',$$

where $B'' \subset B'$ are concentric open balls. If $U_{B'}$ is not trivial, then $U_{B'}$ is positive and the inequality is strict unless U and $U_{B'}$ are equals.

Proof. Let (t_k) be a sequence of positive reals s.t. $t_k \downarrow 0$, $f_k(t) := f(t + t_k + m)$ and $a_k(t, z) := \tilde{a}(t + t_k + m, z)$. Since a_k and f_k satisfy (H1)-(H6) and $f_k(0) = f(t_k + m) > 0$, we can apply Theorem 4.1 to obtain the maximal solution $U_k \in W_0^{1,p}(B')$ of

$$-\operatorname{div} a_k(v, \nabla v) = f_k(v) \quad (4.6)$$

in B' . Observe that if U is a radial solution of $(\tilde{P}_{B''})$ satisfying $0 \leq U \leq m$, then $U - t_k - m \leq 0$ or $U - t_k - m$ is also a solution of (4.6) in a ball contained in B'' vanishing on the boundary of this ball. Then, $U_k > U - t_k - m$. Furthermore, since the important constants $(n, q, \alpha', \beta', C_*, |\Omega|)$ associated a_k and f_k can be chosen not depending k , U_k is bounded in the L^∞ norm by the same argument as in Theorem 4.1. Therefore, following the estimates of Remark 4.1 we get that ∇U_k is a family of equicontinuous functions. Hence, for some subsequence that we denote by U_k , it follows that U_k converges to some function U_0 in the C^1 norm. Therefore, $U_{B'} := U_0 + m$ is a solution of $(\tilde{P}_{B'})$, with $U_{B'} = m$ on $\partial B'$, and $U_{B'} \geq U$, proving the first part.

Suppose now that $U_{B'}$ is not trivial. According to Remark 4.1, $U_{B'} = w_0(|x|)$ for some nonnegative nonincreasing function $w_0 : [0, R_0] \rightarrow \mathbb{R}$. If $w_0(r^*) = 0$ for some $r^* \in [0, R_0]$, then $w'(r^*) = 0$ since w is differentiable. But, this contradicts Lemma 2.4 and the fact that $f(U_{B'})$ is positive in some nontrivial set. Then $U_{B'}$ is positive in B' . If U is a radial solution in B'' different from $U_{B'}$, then these functions cannot be equal at some point, otherwise U touches $U_{B'}$ by below contradicting the uniqueness of solution for ODE. \square

Remark 4.2. *If (H6) is not satisfied in Theorem 4.1 or 4.2, we still have the existence of U_B such that $U_B \geq U$, as we will see in the next section as a*

particular case of the main theorem. However, we cannot guarantee the strict inequality. Maybe it is possible that $U_B(0) = U(0)$ and $U_B \not\equiv U$, since (H6) is important for uniqueness of solution for (4.3).

5 Estimates for sublinear equations

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, B be the ball centered at the origin with $|B| = |\Omega|$, and suppose that a and f satisfy hypotheses (H1)-(H5) and \tilde{a} satisfies (H3)-(H4), possibly with different constants $(\tilde{C}_s, \tilde{C}_*, \tilde{C}^*)$ and different powers $(\tilde{p}, \tilde{q}, \tilde{q}_0)$. Assume also that $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ for any $z \in \mathbb{R}^n$ and $\tilde{a}(t, z) \cdot z = \tilde{C}_s |z|^{\tilde{q}_0}$ for $|z| < \delta$, where $\delta \in (0, 1)$. Then, there exists a radial solution $U_B \in W_0^{1,p}(B)$ of (\tilde{P}_B) s.t. for any solution u of (P_Ω) ,*

$$U_B \geq u^\sharp \quad \text{in } \Omega^\sharp.$$

Remark 5.1. *There exists a function $a^*(z) \in C^0(\mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$, of the form $a^*(z) = b^*(|z|)z/|z|$, where $b^* \in C^1(\mathbb{R} \setminus \{0\})$ is positive on $\mathbb{R} \setminus \{0\}$, $a^*(0) = 0$, $a^*(z) \cdot z$ is convex, that satisfies*

- $|a^*| \leq |\tilde{a}|$,
- $a^*(z) \cdot z = \tilde{C}_s |z|^{\tilde{q}_0}$ for $|z| < \delta$,
- $a^*(z) \cdot z \geq \eta \tilde{C}_* |z|^{\tilde{q}}$ for $|z| \geq 1$, where $\eta \in (0, 1)$,
- $a^*(z) \cdot z = \eta \tilde{C}_* |z|^{\tilde{q}}$ for z large.

For that, define b^ in $[0, \delta]$ by $b^*(s) = \tilde{C}_s s^{\tilde{q}_0-1}$. Then, extend $s b^*(s)$ linearly to $[\delta, 1]$ in such a way that it is C^1 in $[0, 1]$. Defining $a^*(z) = b^*(|z|)z/|z|$, we have that $|a^*| \leq |\tilde{a}|$ in $B_1(0)$ from the convexity of $\tilde{a}(t, z) \cdot z$. Let $h = b^*(1)$ and $\eta' < \min\{1, h/C^*\}$. Hence, $s b^*(s)|_{s=1} > \eta' \tilde{C}_* s^{\tilde{q}}|_{s=1}$ and we can extend $s b^*(s)$ linearly until the graph $(s, s b^*(s))$ reaches $(s, \eta' \tilde{C}_* s^{\tilde{q}})$ at some point s_0 . So define $b^*(s)$ that satisfies $s b^*(s) < \eta' \tilde{C}_* s^{\tilde{q}}$ for $s > s_0$, $s b^*(s)$ is convex and $s b^*(s) = \eta' \tilde{C}_* s^{\tilde{q}}/2$ for s large. Taking $\eta = \eta'/2$, the function $a^*(z)$ defined from b^* as before, fulfills the requirements.*

Lemma 5.1. *Assume the same hypotheses as in the previous proposition and that u is a solution of (P_Ω) . Then there exists $t_0 \leq \sup u$, an open ball B^* centered at 0 with the same measure as $\{u \geq t_0\}$, and a radial solution U_{t_0} for*

$$\begin{cases} -\operatorname{div} \tilde{a}(V, \nabla V) &= f(V) & \text{in } B^* \\ V &= t_0 & \text{on } \partial B^* \end{cases} \quad (5.1)$$

such that $U_{t_0} \geq u^\sharp$ in B^ .*

Proof. Let $M = \operatorname{ess\,sup} u > 0$, that is finite by Lemma 2.2.

Possibility 1: $|\{u = M\}| > 0$

Let r_0 be such that the ball $B^* = B_{r_0}(0)$ has the same measure as $\{u = M\}$. Applying Theorem 4.1 or Theorem 4.2 for $B' = B_{r_0}$ and $m = M$, there exists some maximal solution $U_{B'}$ for (5.1) with $t_0 = M$. Then, the result follows

taking $t_0 = M$ and $U_{t_0}(x) = U_{B'}$.

Possibility 2: $|\{u = M\}| = 0$

Since f is locally Lipschitz and positive in some neighborhood of M , there exists some $\varepsilon_0 > 0$ such that, for any $\varepsilon \leq \varepsilon_0$, the function

$$G_\varepsilon(t) := \frac{f(t)}{(t - (M - \varepsilon))^{\tilde{q}_0 - 1}}$$

is decreasing on $(M - \varepsilon, M + \varepsilon_0)$.

Part 1: For $\varepsilon' \leq \varepsilon_0$ small and $t_1 \in (M - \varepsilon', M)$, there is a solution U_{t_1} to the problem (5.1) with t_0 replaced by t_1 such that $|\{U_{t_1} > t_1\}| = \mu_u(t_1)$, $\sup U_{t_1} < M + \varepsilon_0$ and $|\nabla U_{t_1}| \leq \delta$, where δ is given in Proposition 5.1.

To prove this, observe that the definition of M implies that $\mu_u(t) > 0$ for $t \in (M - \varepsilon_0, M)$. For $t_1 \in (M - \varepsilon_0, M)$, let r_1 be such that the ball $B_{r_1}(0)$ satisfies $|B_{r_1}(0)| = \mu_u(t_1)$. Using the same argument as in the Possibility 1, there exists a radial solution U_{t_1} for (5.1) with t_0 and B_{r_0} replaced by t_1 and B_{r_1} . We have that $U_{t_1} - t_1$ is a solution of

$$-\operatorname{div} \bar{a}(U, \nabla U) = \tilde{f}(U),$$

where $\bar{a}(t, z) = \tilde{a}(t + t_1, z)$ and $\tilde{f}(t) = f(t + t_1)$, that vanishes on $\partial B_{r_1}(0)$. Note that \bar{a} and \tilde{f} satisfy (H1)-(H6) (the constants associated to \tilde{f} are $\alpha' \in (\alpha, \tilde{C}_* \lambda_B)$ and β' as in the proof of Theorem 4.1). Hence, (2.3) implies that $\sup U_{t_1} - t_1 \leq C|B_{r_1}(0)|^\sigma$, where $C = C(n, \tilde{q}, \alpha', \beta', \frac{\eta \tilde{C}_*}{\tilde{q}_0}, |\Omega|) > 0$, η is associated to a^* from Remark 5.1, and $\sigma = 1/q$ if $q > n$ or $\sigma = 1/n$ if $q \leq n$. (Since $\eta \in (0, 1)$ and $\tilde{q}_0 > 1$, any operator \bar{a} satisfying $\bar{a}(t, z) \cdot z \geq \tilde{C}_* |z|^{\tilde{q}}$ also satisfies $\bar{a}(t, z) \cdot z \geq \frac{\eta \tilde{C}_*}{\tilde{q}_0} |z|^{\tilde{q}}$. Thus we can consider $C = C(n, \tilde{q}, \alpha', \beta', \frac{\eta \tilde{C}_*}{\tilde{q}_0}, |\Omega|) \geq C_1 := C_1(n, \tilde{q}, \alpha', \beta', \tilde{C}_*, |\Omega|)$ and we can take C instead C_1 .) Therefore,

$$\sup U_{t_1} \leq C(\mu_u(t_1))^\sigma + t_1 \leq C(\mu_u(t_1))^\sigma + M.$$

For $\varepsilon_1 \leq \varepsilon_0$ that will be defined later, since

$$\lim_{t \rightarrow M^-} \mu_u(t) = |\{u = M\}| = 0,$$

we get $(\mu_u(t))^\sigma < \varepsilon_1/C$ for $t \in (M - \varepsilon', M)$, where $\varepsilon' \leq \varepsilon_0$ is small enough. Thus, $\sup U_{t_1} < M + \varepsilon_0$. For $t \geq t_1$, define $r(t)$ such that $\partial B_{r(t)}(0) = \{U_{t_1} = t\}$. Then, in the case $|\nabla U_{t_1}(x)| \leq 1$, (H4) and Lemma 2.4 imply that

$$\begin{aligned} n\omega_n r(t)^{n-1} \tilde{C}_s |\nabla U_{t_1}(x)|^{\tilde{q}_0} &\leq \int_{\partial B_{r(t)}} |\tilde{a}(U_{t_1}, \nabla U_{t_1})| dH^{n-1} \\ &= \int_{B_{r(t)}(0)} f(U_{t_1}) dx \leq \omega_n r(t)^n f(M + \varepsilon_0), \end{aligned}$$

for $x \in \{U_{t_1} = t\}$. From this estimate and $|B_{r(t)}| \leq |B_{r_1}| = \mu_u(t_1) < (\varepsilon_1/C)^{\frac{1}{\sigma}}$,

$$|\nabla U_{t_1}(x)| \leq \left(\frac{\varepsilon_1}{C\omega_n^\sigma} \right)^{\frac{1}{\sigma n \tilde{q}_0}} \left(\frac{f(M + \varepsilon_0)}{n\tilde{C}_s} \right)^{\frac{1}{\tilde{q}_0}} \quad \text{for } x \in B_{r_1}(0).$$

In the case $|\nabla U_{t_1}(x)| > 1$, a similar estimate holds replacing \tilde{C}_s by \tilde{C}_* and \tilde{q}_0 by \tilde{q} . Any way, taking ε_1 small, $|\nabla U_{t_1}(x)| \leq \delta$, where δ is given in hypothesis of Proposition 5.1. Therefore, U_{t_1} satisfies the \tilde{q}_0 laplacian equation

$$-\tilde{C}_s \Delta_{\tilde{q}_0} U_{t_1} = f(U_{t_1}) \quad \text{in } B_{r_1}. \quad (5.2)$$

Part 2: U_{t_1} is the minimizer of the functional

$$I_{t_1}(V) := \int_{B_{r_1}} \frac{\nabla V \cdot \tilde{a}(V, \nabla V)}{\tilde{q}_0} - \bar{F}(V) \, dx$$

in the space $E = \{V \in W^{1,\tilde{q}}(B_{r_1}) \mid V = t_1 \text{ on } \partial B_{r_1}\}$, where $\bar{F}(t) = \int_0^t \bar{f}(s) ds$,

$$\bar{f}(s) = \begin{cases} f(s) & \text{if } s \leq M + \varepsilon_0 \\ f(M + \varepsilon_0) & \text{if } s > M + \varepsilon_0. \end{cases}$$

For that, consider a^* with the properties stated in the Remark 5.1. Therefore,

$$I_{t_1}^*(V) \leq I_{t_1}(V) \quad \text{for } V \in E,$$

where $I_{t_1}^*$ is defined replacing \tilde{a} by a^* in the definition of I_{t_1} . From the growth conditions on a^* and \bar{f} , we can use standarts techniques to prove that $I_{t_1}^*$ has a global minimum $U^* \in E$. Moreover, this minimum is a solution of

$$-\operatorname{div} \hat{a}(\nabla V) = \bar{f}(V) \quad \text{in } B_{r_1},$$

where

$$\hat{a}(z) := \frac{a^*(z) + z \cdot Da^*(z)}{\tilde{q}_0}.$$

Observe that $\hat{a}(z) \cdot z \geq a^*(z) \cdot z / \tilde{q}_0$ since $s \mapsto |a^*(sz)|$ is increasing from (H3). Hence \hat{a} and \bar{f} satisfy (H1), (H5), $\eta \tilde{C}_* / \tilde{q}_0 (|z|^q - 1) \leq \hat{a}(z) \cdot z$ for $z \in \mathbb{R}^n$, $t \in \mathbb{R}$ where the important constants in order to apply (2.3) are n , \tilde{q} , α , β , $\eta \tilde{C}_* / \tilde{q}_0$ and $|\Omega|$. Then, as in Part 1, $\sup U^* - t_1 < C|B_{r_1}(0)|$, where $C = C(n, \tilde{q}, \alpha', \beta', \frac{\eta \tilde{C}_*}{\tilde{q}_0}, |\Omega|)$ is the same constant as before. (Now it is clear why we chose a constant C depending on $\eta \tilde{C}_* / \tilde{q}_0$ instead of \tilde{C}_* at that moment.) Thus $\sup U^* < M + \varepsilon_0$ and, following the same computations as before, $|\nabla U^*| < \delta$. Then, from $a^*(t, z) = \tilde{a}(t, z)$ for $|z| < \delta$, it follows that

$$I_{t_1}^*(U^*) = I_{t_1}(U^*)$$

and, therefore, U^* is also a global minimizer of I_{t_1} . From $a^*(t, z) = \tilde{C}_s |z|^{\tilde{q}_0-2} z$ for $|z| \leq \delta$, we have that U^* is also a solution of (5.2). Hence $U_{t_1} - t_1$ and $U^* - t_1$ are solutions of $-\tilde{C}_s \Delta_{\tilde{q}_0} U = \tilde{f}(U)$. Taking $\varepsilon = M - t_1$, we have that $\tilde{f}(t)/t^{\tilde{q}_0} = G_\varepsilon(t + t_1)$ that is decreasing on $(M - \varepsilon, M + \varepsilon_0)$ that contains the range of $U_{t_1} - t_1$ and $U^* - t_1$. From the uniqueness result of [13], $U_{t_1} = U^*$.

Part 3: For $t_1 \in (M - \varepsilon', M)$, there exists $t_0 \geq t_1$ and a solution U of (5.1) s.t. $U \geq u^\sharp$ in $B_{r(t_0)} := \{u^\sharp > t_0\}$, $U = u^\sharp$ on $\partial B_{r(t_0)}$ and $|\{U > t_0\}| = |\{u^\sharp > t_0\}|$. Using the properties for Schwarz symmetrization stated in Remark 2.1, the relations $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ and $\bar{F}(u^\sharp) = F(u^\sharp)$, and that U_{t_1} minimizes I_{t_1} ,

$$\begin{aligned} \int_{\Omega_{t_1}} \frac{\nabla u \cdot a(u, \nabla u)}{\tilde{q}_0} - F(u) dx &\geq \int_{B_{r_1}} \frac{\nabla u^\sharp \cdot a(u^\sharp, \nabla u^\sharp)}{\tilde{q}_0} - F(u^\sharp) dx \\ &\geq \int_{B_{r_1}} \frac{\nabla u^\sharp \cdot \tilde{a}(u^\sharp, \nabla u^\sharp)}{\tilde{q}_0} - \bar{F}(u^\sharp) dx \\ &\geq \int_{B_{r_1}} \frac{\nabla U_{t_1} \cdot \tilde{a}(U_{t_1}, \nabla U_{t_1})}{\tilde{q}_0} - \bar{F}(U_{t_1}) dx. \end{aligned}$$

Hence, from Lemma 2.3 and $\bar{F}(U_{t_1}) = F(U_{t_1})$, we have

$$\int_{\Omega_{t_1}} \frac{(u - t_1)f(u)}{\tilde{q}_0} - F(u) dx \geq \int_{B_{r_1}} \frac{(U_{t_1} - t_1)f(U_{t_1})}{\tilde{q}_0} - F(U_{t_1}) dx,$$

that is equal to estimate (3.5). Note also that

$$h_{t_1}(s) = \frac{(s - t_1)f(s)}{\tilde{q}_0} - F(s)$$

is decreasing in $(t_1, M + \varepsilon_0)$ since $G_\varepsilon(s)$ is decreasing in this interval, where $\varepsilon = M - t_1 < \varepsilon_0$. Therefore, using that $U_{t_1}(B_{r_1}), u(B_{r_1}) \subset [t_1, M + \varepsilon_0)$ and an argument similar to the one that come after (3.5), we have

$$\max u \leq \max U_{t_1}.$$

If $u^\sharp \leq U_{t_1}$ in B_{r_1} , Part 3 is proved taking $t_0 = t_1$. Otherwise, there exist $t_2 \in (t_1, M)$ such that $\mu_u(t_2) > \mu_{U_{t_1}}(t_2)$. Therefore $B' = \{u^\sharp > t_2\}$ and $B'' = \{U_{t_1} > t_2\}$ are concentric balls satisfying $|B'| > |B''|$. Hence, from Theorem 4.1 or 4.2, there exists some solution U_{t_2} of (5.1) with t_0 replaced by t_2 , such that $\{U_{t_2} > t_2\} = B'$ and $U_{t_2} > U_{t_1}$ in B'' . Since

$$\max u^\sharp \leq \max U_{t_1} < \max U_{t_2},$$

it follows from the right continuity of μ_u and the continuity of $\mu_{U_{t_2}}$ that there exists $t_0 \geq t_2$, such that $|\{U_{t_2} > t_0\}| = |\{u^\sharp > t_0\}|$ and $U_{t_2} \geq u^\sharp$ in $\{u^\sharp > t_0\}$, proving this part.

Part 4: There exists a solution U_{t_0} of (5.1) s.t. $U_{t_0} \geq u^\sharp$ in $B^* := \{u^\sharp \geq t_0\}$, $U = u^\sharp$ on ∂B^* and $|\{U_{t_0} \geq t_0\}| = |\{u^\sharp \geq t_0\}|$.

Let t_0 and U as in Part 3. If $|\{u^\sharp \geq t_0\}| = \mu_u(t_0)$, then the theorem is proved with $B^* = \{u^\sharp \geq t_0\}$. Otherwise, applying Theorem 4.1 or 4.2 for $B' = \{u^\sharp \geq t_0\}$ and $B'' = \{u^\sharp > t_0\}$, there exists a solution U_{t_0} of (5.1) s.t. $U_{t_0} > U$ in B'' , proving the result with $B^* = B'$. \square

Now we present a result that resemble a maximum principle for distribution function in the sense that the distribution μ_u of a solution cannot touch by below the distribution μ_U of a radial solution if $\mu_u \leq \mu_U$.

Proposition 5.2. *Suppose that a , \tilde{a} and f satisfy (H2)-(H4), where the constants and powers presented in (H4) associated to \tilde{a} are given by $(\tilde{C}_s, \tilde{C}_*, \tilde{C}^*)$ and $(\tilde{p}, \tilde{q}, \tilde{q}_0)$, and that $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ for any $z \in \mathbb{R}^n$. Assume also that $u \in W_0^{1,p}(\Omega)$ is a solution of (P_Ω) and $U \in W^{1,p}(B) \cap C^1(B)$ is a radial solution of (\tilde{P}_B) that not necessarily vanishes on ∂B . If $u^\sharp \leq U$ and $u^\sharp \neq U$, then there exists $t_1 \geq 0$ such that $u^\sharp < U$ in $\{U > t_1\}$ and $u^\sharp = U$ in $\{U \leq t_1\}$.*

Moreover, assuming that $u^\sharp \leq U$, if f is strictly increasing and $u^\sharp \neq U$, or Ω is not a ball and $a = a(z)$ (or $\tilde{a} = \tilde{a}(z)$) satisfies hypotheses of Proposition 2.1, then $u^\sharp < U$ in B .

Proof. Since $U \geq u^\sharp$ and f is nondecreasing, we have

$$\int_{\{U > t\}} f(U) dx \geq \int_{\{u^\sharp > t\}} f(u^\sharp) dx = \int_{\{u > t\}} f(u) dx, \quad (5.3)$$

for any $t \geq 0$. Hence, applying Lemma 2.4 for u and U and Pólya-Szegő principle, we get

$$\int_{\{U=t\}} \frac{\tilde{a}(U, \nabla U) \cdot \nabla U}{|\nabla U|} dH^{n-1} \geq \int_{\{u^\sharp=t\}} \frac{a(u^\sharp, \nabla u^\sharp) \cdot \nabla u^\sharp}{|\nabla u^\sharp|} dH^{n-1} \quad (5.4)$$

for almost every $t \geq \inf U$. Since $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$, we have the same inequality with a or \tilde{a} appearing in both sides. Letting $r_1 = (\mu_u(t)/\omega_n)^{1/n}$ and $r_2 = (\mu_U(t)/\omega_n)^{1/n}$ we have some t_0 such that $r_1(t_0) < r_2(t_0)$ since $u^\sharp \neq U$. Hence, Lemma 2.5 implies that $u^\sharp < U$ on $\{U > t_0\}$. Indeed, we can infer that the set of t 's, for which $r_1(t) = r_2(t)$, is an interval that contains 0. Denoting the supremum of this set by t_1 , we have the first part of the result.

Now consider the case f is strictly increasing and $t_1 > 0$. Then we have a strict inequality in (5.3) and, therefore, in (5.4) for any $t \in [0, t_1]$, that contradicts $u^\sharp = U$ in $\{0 \leq U < t_1\}$.

If a is as stated in Proposition 2.1, it follows from $\tilde{a}(z) \cdot z \leq a(z) \cdot z$, (5.3), Lemma 2.4, and Pólya-Szegő principle that

$$\int_{U < t} \nabla U \cdot a(\nabla U) dx \geq \int_{u < t} \nabla u \cdot a(\nabla u) dx \geq \int_{u^\sharp < t} \nabla u^\sharp \cdot a(\nabla u^\sharp) dx,$$

for $t < t_1$. Since $u^\sharp = U$ in $\{U < t_1\}$, the three integrals are equals for $t < t_1$, and therefore, Proposition 2.1 implies that u^\sharp is a translation of u in $\{u < t_1\}$ and Ω is a ball, that is an absurd. Replacing a by \tilde{a} , we see that the same conclusion holds if \tilde{a} satisfies the hypotheses of that proposition. \square

Proof. of Proposition 5.1 Observe that \tilde{a} and f satisfy (H1)-(H5). Furthermore \tilde{a} also satisfy (H6), since $|\tilde{a}(t, z)| = \tilde{C}_s |z|^{\tilde{q}_0 - 1}$ for z small. Then let U_B be the solution stated in Theorem 4.1 or in Theorem 4.2 for $m = 0$. Consider the set

$$A = \{t_0 : \exists \text{ a radial sol. } U_{t_0} \text{ of (5.1) s.t. } U_{t_0} \geq u^\sharp \text{ in } B^* \text{ and } |B^*| = |\{u^\sharp \geq t_0\}|\}.$$

According to the previous lemma this set is not empty. To prove the theorem, it suffices to show that $0 \in A$. For that we prove the following assertions.

Assertion 1: For any positive $t_1 \in A$, there exists $t' \in A$ such that $t' < t_1$.

From the definition of A , there exists a radial solution U_{t_1} of (5.1) greater than or equal to u^\sharp in $\{u^\sharp \geq t_1\}$. Since U_{t_1} is radial, it can be extended as a positive radial solution of $-\operatorname{div}(\tilde{a}(V, \nabla V)) = f(V)$ in some ball that contains $\{u^\sharp \geq t_1\}$ or in \mathbb{R}^n . The maximal extension will be denoted by U_{t_1} . Consider

$$D = \{t \geq 0 : |\{U_{t_1} > t\}| = |\{u \geq t\}| \text{ and } |\{U_{t_1} > s\}| \geq |\Omega_s| \text{ for } s > t\},$$

and let $t_2 = \inf D$. Observe that $t_1 \in D$ and so $t_2 \leq t_1$. If $t_2 < t_1$, then there exists $t_3 \in [t_2, t_1) \cap D$. Hence, in this case, our assertion is proved taking $t' = t_3$. Consider now the case $t_2 = t_1$. Thus $0 \notin D$, since $0 < t_1 = t_2$. Therefore, there are two possibilities:

- 1) $|\{U_{t_1} > 0\}| > |\Omega|$ and $|\{U_{t_1} > s\}| \geq |\Omega_s|$ for $s > 0$;
- 2) $|\{U_{t_1} > s_0\}| < |\Omega_{s_0}|$ for some $s_0 \geq 0$.

Case 1): since $|\{U_{t_1} > s\}| \geq \mu_u(s)$ for $s > 0$, $U_{t_1} \geq u^\sharp$. Then, from the first part of Proposition 5.2, $U_{t_1} = u^\sharp$ in $\{U_{t_1} < t_2\}$, since $U_{t_1} = u^\sharp$ in $\{U_{t_1} = t_2\}$. However, this contradicts $|\{U_{t_1} > 0\}| > |\Omega|$ and, so this case is not possible.

Case 2): from the definition of t_1 , it follows that $s_0 < t_1$. Let $B'_{s_0}(0)$ be a ball such that $|B'_{s_0}| = |\{u \geq s_0\}|$. Hence $B' = B'_{s_0}(0)$ and $B'' = \{U_{t_1} > s_0\}$ satisfy $|B'| > |B''|$ and, from Theorem 4.1 or 4.2, there exists a solution U_{s_0} of $(\tilde{P}_{B'})$ with $U_{s_0} = s_0$ on $\partial B'$, such that $U_{s_0} > U_{t_1}$ in B'' . Then $U_{s_0} > U_{t_1} \geq u^\sharp$ in $\{U_{t_1} > t_1\}$ and, therefore,

$$\mu_{U_{s_0}}(t_1) = |\{U_{s_0} > t_1\}| > |\{U_{t_1} > t_1\}| = |\{u \geq t_1\}| = \mu_u(t_1^-).$$

Since $\mu_{U_{s_0}}$ is continuous and $\mu_u(t_1^-) = \lim_{t \rightarrow t_1^-} \mu_u(t)$, we have $\mu_{U_{s_0}}(t) > \mu_u(t)$ for $s_0 < t < t_1$, sufficiently close to t_1 . Defining

$$t' = \inf\{t \geq s_0 : \mu_{U_{s_0}}(t) > \mu_u(t)\},$$

it follows that $s_0 \leq t' < t_1$ and $\mu_u(t') \leq \mu_{U_{s_0}}(t') \leq \mu_u(t'^-)$. Observe also that $U_{t_1} > u^\sharp$ in $\{u^\sharp > t'\}$. Hence, this assertion is proved if $\mu_{U_{s_0}}(t') = \mu_u(t'^-)$. If $\mu_{U_{s_0}}(t') < \mu_u(t'^-)$, applying Theorem 4.1 or Theorem 4.2 for the balls $\{U_{s_0} > t'\} \subsetneq \{u^\sharp \geq t'\}$, we get a solution $U_{t'}$ s.t. $U_{t'} > U_{s_0}$ in $\{U_{s_0} > t'\}$ and $|\{U_{t'} > t'\}| = |\{u^\sharp \geq t'\}|$. Then $U_{t'} > u^\sharp$ in $\{u^\sharp \geq t'\}$ and $U_{t'} = u^\sharp$ on $\partial\{u^\sharp \geq t'\}$, completing Assertion 1.

Assertion 2: If $t_1 = \inf A$, then $t_1 \in A$.

We can prove this using the same limit argument as in Lemma 4.1.

These assertions imply that $\inf A = 0$. Then there is a solution U_0 of (\tilde{P}_B) such that $U_0 \geq u^\sharp$. Since U_B is maximal, it follows that $U_0 \leq U_B$, proving the result. \square

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, B be a ball centered at the origin with $|B| = |\Omega|$, and suppose that a , \tilde{a} and f satisfy the hypotheses (H1)-(H5), where the constants and powers associated to a and \tilde{a} may be different. If $\tilde{a}(t, z) \cdot z \leq a(t, z) \cdot z$ for any $z \in \mathbb{R}^n$, then there exists a radial solution $U_B \in W_0^{1,p}(B)$ of (\tilde{P}_B) such that*

$$U_B \geq u^\sharp \quad \text{in } B,$$

where u^\sharp is the symmetrization of any solution u of (P_Ω) .

Furthermore, if Ω is not a ball and $a = a(z)$ (or $\tilde{a} = \tilde{a}(z)$) is as stated in Proposition 2.1, then $U_B > u^\sharp$.

Proof. For $k \in \mathbb{N}$, let $a_k(t, z) = b_k(t, |z|)z/|z|$ be a function satisfying (H3) s.t.

- $|a_k| \leq |\tilde{a}|$,
- $a_k(t, z) \cdot z = C|z|^{\tilde{q}_0}$ for some $C > 0$ and $|z| \leq 1/k$,
- $a_k(t, z) \cdot z = \tilde{a}(t, z) \cdot z$ for $|z| \geq 2/k$.

To obtain such a_k , first observe that the convexity of $\tilde{a}(t, z) \cdot z$ in z and the relation $\tilde{a}(t, z) \cdot z \geq \tilde{C}_s|z|^{\tilde{q}_0}$ imply that the derivative of $s \mapsto \tilde{a}(t, sw) \cdot sw$ is uniformly bounded from below by some $D_k > 0$ for $t \in \mathbb{R}$, $|w| = 1$ and $s = 1/k$. From $\tilde{a}(t, z) = \tilde{b}(t, |z|)z/|z|$, we get $\partial_s[\tilde{b}(t, s)s] \geq D_k$ for $s = 1/k$ and $t \in \mathbb{R}$. Since $\tilde{a}(t, z) \cdot z$ in z is convex, $\partial_s[\tilde{b}(t, s)s]$ is increasing in s and, then $\partial_s\tilde{b}(t, s)s \geq D_k$ for $s = 2/k$. Now define $b_k(t, s)$ in $\mathbb{R} \times [0, 1/k]$ by $b_k(t, s) = C_k|s|^{\tilde{q}_0-1}$, where C_k is such that $\partial_s[b_k(t, s)s] = D_k/2$ for $s = 1/k$. (Indeed we can chose $D_k = \tilde{C}_s(1/k)^{\tilde{q}_0-1}$ and $C = C_k = \tilde{C}_s/(2\tilde{q}_0)$.) Hence it is possible to extend b_k to $\mathbb{R} \times [0, +\infty)$ in such a way that $\partial_s[b_k(t, s)s]$ is strictly increasing in s , continuous and $b_k(t, s) = \tilde{b}(t, s)$ for $s \geq 2/k$. The function a_k defined from b_k satisfies the required properties.

Since a , a_k and f satisfy the hypotheses of Proposition 5.1, there exists some radial solution $U_k \in W_0^{1,p}(B)$ of $-\operatorname{div} a_k(V, \nabla V) = f(V)$ in B that satisfies $U_k \geq u^\sharp$, for any solution u of (P_Ω) . Using (2.3), it follows that the sequence (U_k) is bounded in the L^∞ norm and, following the same argument as in Part 1 of Lemma 5.1, the derivative of U_k is also uniformly bounded and equicontinuous. Hence, some subsequence converges to some function U_B that is a weak solution of (\tilde{P}_B) , by standart arguments. Moreover, $U_k \geq u^\sharp$ implies that $U_B \geq u^\sharp$, for any solution u of (P_Ω) , completing the first part of the theorem.

Suppose now that Ω is not a ball and u is a solution of (P_Ω) . From the first part, $U_B \geq u^\sharp$ and, therefore, applying Proposition 5.2, $U_B > u^\sharp$. \square

6 Existence and bound result

First we apply the results of the previous section to prove that the symmetrization of solutions of (1.4) are bounded by a radial solution. Notice that if h is also bounded from above, the proof follows immediately from Theorem 5.1 applied to the equation $-\operatorname{div}(h(v)a(\nabla v)) = f(v)$. For h just bounded from below

by some positive constant, proceed as follows: let $m = \inf h$, $a_0(t, z) = m a(z)$ and $a_1(t, z) = h(t)a(z)$. Since $a_0(t, z) \cdot z \leq a_1(t, z) \cdot z$ and a_0 fulfill all necessary assumptions, Theorem 5.1 implies that there exists a solution U_0 for

$$-m \operatorname{div}(a(\nabla V)) = f(V) \quad \text{in } B,$$

such that $U_0 \geq u^\sharp$, where u is any solution of (1.4). Let $M = \max U_0$, h_1 be a C^1 function such that $h_1(t) = h(t)$ for $t \leq M$ and $h_1(t) = h(M+1)$ for $t \geq M+1$, and $a_2(t, z) = h_1(t)a(z)$. Observe that u is solution of $-\operatorname{div}(a_2(v, \nabla v)) = f(v)$ and a_2 satisfies (H1)-(H5). Hence from Theorem 5.1, there exists a radial solution U_B of $-\operatorname{div}(a_2(V, \nabla V)) = f(V)$ in B such that $U_B \geq u^\sharp$. Moreover $U_B \leq U_0$, since $a_2 \geq a_0$. Therefore U_B is also a solution of $-\operatorname{div}(h(V)a(\nabla V)) = f(V)$ completing the proof. This can be summarized in the next proposition.

Proposition 6.1. *If $a_1 = ha$ and $f = gh$ satisfy (H1)-(H5), then there exists a radial function U_B , solution of (1.4) when the domain is B , such that $U_B \geq u^\sharp$, where u^\sharp is the symmetrization of any solution of (1.4). This is also true if a_1 does not satisfy the right inequality of (1.1).*

This result gives a priori estimate of a solution u , but does not prove its existence, except for the ball where we obtain the function U_B . We show now an existence result for a particular case, using this estimates.

Theorem 6.1. *Let $a(z) = z|z|^{p-2}$ and suppose that $a_1 = ha$ and $f = gh$ satisfy (H1)-(H5), with the possibility of not fulfillment of the right inequality of (1.1). Then there exists a solution u to the problem (1.4).*

Proof. Let M , h_1 and U_B be as defined before. Define the functional

$$J(v) = \int_{\Omega} (h_1(v))^{\frac{p}{p-1}} \frac{|\nabla v|^p}{p} - \int_0^v f(s)(h_1(s))^{\frac{1}{p-1}} ds \, dx.$$

Since h_1 is bounded from above and from bellow by some positive constants, conditions (H4) and (H5) holds with $q = q_0 = p$. Then we can minimize J in $W^{1,p}(\Omega)$ and obtain a solution u to $-\operatorname{div}(h_1(v)v|v|^{p-2}) = f(v)$. From the previous result, we have that u is bounded by U_B and, therefore, is a solution that we are looking for. \square

7 Estimates for Eigenfunctions

In the next result, the estimate (7.2) and (1.6) were established in [19] and [20] for $p = q = 2$, with the best constant, and extended in [3] for $p = q > 1$, when λ is the first eigenvalue.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and w be a solution of*

$$\begin{cases} -\Delta_p v &= \lambda v|v|^{q-2} & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega \end{cases} \quad (7.1)$$

where $1 < q \leq p$ and λ is either a real number if $q < p$ or any eigenvalue of $-\Delta_p$ with trivial boundary data if $q = p$. Then

$$(\max |w|)^{1+\frac{n(p-q)}{rp}} \leq \frac{2}{(\omega_n)^{1/r}} \left(\frac{2(p-1)}{p} \right)^{\frac{n(p-1)}{rp}} \left(\frac{\lambda}{n} \right)^{n/rp} \|w\|_r, \quad (7.2)$$

for any $r > 0$. Furthermore,

$$|\tilde{\Omega}_t| \geq \omega_n (\|w\|_\infty - t)^{\frac{n(p-1)}{p}} \left(\frac{p}{p-1} \right)^{\frac{n(p-1)}{p}} \left(\frac{n}{\lambda} \right)^{n/p} \|w\|_\infty^{\frac{n(1-q)}{p}}, \quad (7.3)$$

where $\tilde{\Omega}_t = \{|w| > t\}$, $t \in [0, \max |w|]$.

Proof. Let $M = \|w\|_\infty$, $\rho \geq 1$, and $\Omega_2 = \{x : |w(x)| > M/\rho\}$. Then

$$\|w\|_r^r = \int_\Omega |w|^r dx \geq \int_{\Omega_2} |w|^r dx \geq \left(\frac{M}{\rho} \right)^r |\Omega_2| \quad (7.4)$$

On the other hand,

$$-\Delta_p w = \lambda w |w|^{q-2} \leq \lambda M^{q-1}$$

Hence, by the comparison principle of [22], $|w| \leq u$ in Ω_2 , where u is solution of

$$\begin{cases} -\Delta_p v &= \lambda M^{q-1} & \text{in } \Omega_2 \\ v &= \frac{M}{\rho} & \text{on } \partial\Omega_2 \end{cases}$$

Let U be the solution of

$$\begin{cases} -\Delta_p V &= \lambda M^{q-1} & \text{in } B \\ V &= \frac{M}{\rho} & \text{on } \partial B \end{cases}$$

where B is a ball such that $|B| = |\Omega_2|$. From Theorem 1 of [43] or Theorem 3.2, $u^\sharp \leq U$. Then

$$M = \max |w| \leq \max u = \max u^\sharp \leq \max U$$

We can compute U explicitly:

$$U(x) = \frac{p-1}{p} \left(\frac{\lambda}{n} \right)^{\frac{1}{p-1}} M^{\frac{q-1}{p-1}} \left(R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}} \right) + \frac{M}{\rho},$$

where $\omega_n R^n = |\Omega_2| = |B|$. Since $M \leq \max U = U(0)$,

$$M \leq \frac{p-1}{p} \left(\frac{\lambda}{n} \right)^{\frac{1}{p-1}} M^{\frac{q-1}{p-1}} R^{\frac{p}{p-1}} + \frac{M}{\rho}.$$

Hence,

$$R \geq \left[\frac{(\rho-1)p}{\rho(p-1)} \right]^{\frac{p-1}{p}} \left(\frac{n}{\lambda} \right)^{1/p} M^{\frac{p-q}{p}}.$$

Using this and $R = \left(\frac{|\Omega_2|}{\omega_n}\right)^{1/n}$, we get

$$|\Omega_2| \geq \omega_n \left[\frac{(\rho-1)p}{\rho(p-1)} \right]^{\frac{n(p-1)}{p}} \left(\frac{n}{\lambda} \right)^{n/p} M^{\frac{n(p-q)}{p}}.$$

From this, we get the estimate for $|\Omega_t|$ taking $t = M/\rho$. Moreover applying this inequality with $\rho = 2$ and using (7.4), it follows that

$$\|w\|_r^r \geq \frac{1}{2^r} \omega_n \left(\frac{p}{2(p-1)} \right)^{\frac{n(p-1)}{p}} \left(\frac{n}{\lambda} \right)^{n/p} M^{\frac{n(p-q)}{p} + r}.$$

□

Remark 7.1. *The estimates of this theorem still holds if $|\Delta_p w| \leq |\lambda w|^{q-2}|w|$ or, equivalently, $-\Delta_p w = \lambda g(w)$, where $|g(w)| \leq |w|^{q-1}$. Hence, using the interpolation inequality,*

$$\|w\|_s \leq \|w\|_\infty^{1-r/s} \|w\|_r^{r/s}, \quad \text{for } 0 < r < s \leq \infty$$

we get (1.6) for solutions of $-\Delta_p w = \lambda g(w)$, where $|g(w)| \leq |w|^{q-1}$, with the boundary condition $w = 0$ on $\partial\Omega$. Inequality (7.2) is also true for solutions of $\operatorname{div}(a(x, Dw)) \leq |\lambda g(w)|$, provided $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is such that some comparison principle holds. For instance, consider the following hypotheses on a given by [22]:

$$\begin{aligned} a &\in C(\bar{\Omega} \times \mathbb{R}^n; \mathbb{R}^n) \cap C^1(\bar{\Omega} \times (\mathbb{R}^n \setminus \{0\}); \mathbb{R}^n), \\ a(x, 0) &= 0 \quad \text{for } x \in \Omega, \\ \langle D_z a(x, z), \xi \rangle &\geq (p-1)|z|^{p-2}|\xi|^2 \quad \text{for } (x, z) \in \Omega \times \mathbb{R}^n \setminus \{0\}, \\ |D_z a(x, z)| &\leq C|z|^{p-2} \quad \text{for } (x, z) \in \Omega \times \mathbb{R}^n \setminus \{0\}, \quad C > 0. \end{aligned} \tag{7.5}$$

Theorem 7.2. *Let w be a bounded solution of*

$$\begin{cases} -\operatorname{div}(a(x, \nabla v)) &= f(v) & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega, \end{cases} \tag{7.6}$$

where a satisfies (7.5) and $f \in C^1(\mathbb{R})$ satisfies $|f(t)| \leq c|t|^{q-1} + d$, with $0 < q \leq p$ and $c, d \geq 0$. Then

$$\|w\|_\infty \leq \max \left\{ C_1 \|w\|_r^{\frac{rp}{n(p-q)+rp}}, C_2 \|w\|_r^{\frac{rp}{n(p-1)+rp}} \right\},$$

where $C_1 = C_1(n, p, q, r, \rho, c)$ and $C_2 = C_2(n, p, r, \rho, d)$ are positive constants.

Proof. We use the same ideas of the last theorem. By the comparison principle of [22], $|w| \leq u$, where u solves $-\operatorname{div}(a(x, \nabla v)) = cM^{q-1} + d$ in Ω_2 and $u = M/\rho$ on $\partial\Omega_2$. Since the hypotheses on a imply that $\langle a(x, z), z \rangle \geq |z|^p$, using the same argument as in Remark 3.3, we have that $\max u \leq \max U$, where U is the solution of $-\Delta_p v = cM^{q-1} + d$ on B and $v = M/\rho$ on ∂B . Notice that U is given by

$$U(x) = \frac{p-1}{p} \left(\frac{1}{n} \right)^{\frac{1}{p-1}} (cM^{q-1} + d)^{\frac{1}{p-1}} \left(R^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}} \right) + \frac{M}{\rho}.$$

Following the same computations as before, we conclude the proof where the constants are given by

$$C_1 = (2c)^{\frac{n}{n(p-q)+rp}} K^{\frac{p}{n(p-q)+rp}} \quad , \quad C_2 = (2d)^{\frac{n}{n(p-1)+rp}} K^{\frac{p}{n(p-1)+rp}},$$

and

$$K = \frac{1}{\omega_n} \left(1 - \frac{1}{\rho} \right)^{-\frac{n(p-1)}{p}} \rho^r \left(\frac{p-1}{p} \right)^{\frac{n(p-1)}{p}} \left(\frac{1}{n} \right)^{\frac{n}{p}}.$$

□

Using the interpolation inequality observed in Remark 7.1, we can obtain estimates for $\|w\|_s$, where $s \in (r, \infty]$.

Now, we use this theorem to show that the L^p norms of a solution goes to zero when its domain becomes “far away” from a ball with the same measure. More precisely, when the first eigenvalue of a domain of a given measure is large, then its L^p norms are small.

Corollary 7.1. *Assuming the same hypotheses as in the previous theorem, if $p = q$ and $c < \lambda_p(\Omega)$, the first eigenvalue of $-\Delta_p$, then*

$$\|w\|_\infty \leq \max \left\{ C_1 \left(\frac{d}{\lambda_p(\Omega) - c} \right)^{\frac{1}{p-1}} |\Omega|^{\frac{1}{p}}, C_2 \left(\frac{d}{\lambda_p(\Omega) - c} \right)^{\frac{\kappa_1}{p-1}} |\Omega|^{\frac{\kappa_1}{p}} \right\},$$

where $\kappa_1 = p^2/[n(p-1) + p^2]$. If $p > q$, then

$$\|w\|_\infty \leq \max \left\{ C_1 \tau^{\frac{rp}{n(p-q)+rp}}, C_2 \tau^{\frac{rp}{n(p-1)+rp}} \right\},$$

where $\tau = |\Omega|^{1/p} \max\{(2c/\lambda_p(\Omega))^{1/(p-q)}, (2d/\lambda_p(\Omega))^{1/(p-1)}\}$.

Proof. First note that the growth condition on f and Hölder inequality imply

$$\lambda_p(\Omega) \|w\|_p^p \leq \|\nabla w\|_p^p \leq \int_\Omega \nabla w \cdot a(\nabla w, x) dx \leq c \|w\|_p^q |\Omega|^{\frac{p-q}{p}} + d \|w\|_p |\Omega|^{\frac{p-1}{p}}.$$

The proof for the case $p = q$ follows directly from this and Theorem 7.2. In the case $p > q$, we get from this inequality that $\|w\|_p \leq \tau$. Hence we complete the proof applying Theorem 7.2. □

Corollary 7.2. *Assume the same hypotheses about a and f as in the previous theorem. Suppose also that $a = a(z)$, $f(t) > 0$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$. If $\lambda_p(\Omega)$ is sufficiently large, then any solution u of (7.6) in Ω satisfies $u^\# < U$, where $u^\#$ is the symmetrization of u and U is the maximal solution of (7.6) in the ball B with the same measure as Ω .*

The novelty in this corollary is that f does not need to be monotone.

Proof. From Hopf lemma, $\partial_n U = c < 0$ on ∂B and, therefore, there exists some “paraboloid”

$$P(x) = \frac{p-1}{p} \left(\frac{1}{n} \right)^{\frac{1}{p-1}} C^{\frac{1}{p-1}} \left(r^{\frac{p}{p-1}} - |x - x_0|^{\frac{p}{p-1}} \right),$$

where x_0 is the center of B and r is its radius, such that $0 < P < U$ in B . Observe that $-\Delta_p P = C$. Since f is continuous and $f(0) = 0$, let $M > 0$ be such that $f(t) < C$ for $t < M$. Corollary 7.1 implies that $\|u\|_\infty < M$, where u is any solution of (7.6), if $\lambda_p(\Omega)$ is large enough. Then

$$-\operatorname{div} a(\nabla u) = f(u) \leq C = -\Delta_p U,$$

and, from Theorem 5.1, $u^\# \leq P < U$ proving the result. \square

8 Appendix

We show now Lemma 2.2 with the same arguments as in Theorem 3.11 of [38].

Proof. of Lemma 2.2 Let $K > 0$, $\ell > 0$, $r \geq 1$, $\gamma = qr - q + 1$,

$$v = P(u) = \min\{(u + K)^r, \ell^{r-1}(u + K)\}$$

and

$$\varphi = G(u) = \min\{(u + K)^\gamma, \ell^{\gamma-1}(u + K)\} - K^\gamma \in W_0^{1,q}(\Omega').$$

Then, using that $a(t, z) \cdot z \geq C_*(|z|^q - 1)$ for all $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we get

$$\begin{aligned} \int_{\Omega'} |\nabla v|^q dx &\leq \int_{\Omega'} |P'(u)|^q \left(\frac{\nabla u \cdot a(u, \nabla u)}{C_*} + 1 \right) dx \\ &\leq \int_{\Omega'} \frac{|P'(u)|^q}{G'(u)} \cdot \frac{\nabla \varphi \cdot a(u, \nabla u)}{C_*} dx + \int_{\Omega'} |P'(u)|^q dx. \end{aligned}$$

Notice that $|P'(u)|^q/G'(u) = E$, where $E = 1$ if $u + K > \ell$ and $E = r^q/\gamma$ if $u + K < \ell$. Then, $E \leq r^q$ and, using $\nabla \varphi \cdot a(u, \nabla u) \geq 0$,

$$\begin{aligned} \int_{\Omega'} |\nabla v|^q dx &\leq \frac{r^q}{C_*} \int_{\Omega'} \nabla \varphi \cdot a(u, \nabla u) dx + \int_{\Omega'} |P'(u)|^q dx \\ &= \frac{r^q}{C_*} \int_{\Omega'} f(u) G(u) dx + \int_{\Omega'} |P'(u)|^q dx. \end{aligned} \tag{8.1}$$

Observe now that, for $u + K < \ell$,

$$\begin{aligned} f(u)G(u) &\leq (\alpha u^{q-1} + \beta) \cdot (u + K)^\gamma \leq \alpha(u + K)^{q-1+\gamma} + \beta \frac{(u + K)^{q-1+\gamma}}{K^{q-1}} \\ &\leq v^q \left(\alpha + \frac{\beta}{K^{q-1}} \right). \end{aligned}$$

In a similar way, we can prove this inequality also for the case for $u + K \geq \ell$. Furthermore, for $u + K \leq \ell$,

$$|P'(u)|^q = |r(u + K)^{r-1}|^q = r^q \frac{(u + K)^{rq}}{(u + K)^q} \leq r^q \frac{v^q}{K^q},$$

that is also true for $u + K > \ell$. From these two inequalities and (8.1), we get

$$\int_{\Omega'} |\nabla v|^q dx \leq \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{r^q}{K^q} \right] \int_{\Omega'} v^q dx. \quad (8.2)$$

Now we study the cases $q > n$, $q < n$ and $q = n$ separately.

Case 1: $q > n$.

Observe that for $r = 1$, we get $v = u + K$. Using the Morrey's inequality for $v - K \in W_0^{1,q}$,

$$\|v - K\|_{C^{0,1-n/q}} \leq \tilde{C}_0 \|v - K\|_{W^{1,q}} \leq C_0 (\|v\|_q + K|\Omega'|^{1/q} + \|Dv\|_q),$$

where $C_0 = C_0(n, q)$. From this one and (8.2), we get

$$\sup u = \sup v - K \leq \left[C_0 + \left[\frac{1}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{1}{K^q} \right]^{1/q} \right] \|v\|_q + C_0 K |\Omega'|^{1/q}.$$

Since $\|v\|_q = \|u + K\|_q \leq \|u\|_q + K|\Omega'|^{1/q}$, we get $\sup u \leq D_1 \|u\|_q + D_2 K |\Omega'|^{1/q}$, where

$$D_1 = C_0 + \left[\frac{1}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{1}{K^q} \right]^{1/q} \quad \text{and} \quad D_2 = D_1 + C_0.$$

Case 2: $q < n$.

Since $v - K^r \in W_0^{1,q}(\Omega')$, the Sobolev inequality implies

$$\|v - K^r\|_{q^*} \leq C_0 \|\nabla v\|_q, \quad (8.3)$$

where $q^* = nq/(n - q)$ and $C_0 = \frac{q(n-1)}{n-q}$. Using this and (8.2), we get

$$\|v - K^r\|_{q^*} \leq C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{r^q}{K^q} \right]^{1/q} \|v\|_q. \quad (8.4)$$

Hence, naming $\chi = n/(n - q)$, it follows that

$$\|v\|_{\chi q} \leq C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{r^q}{K^q} \right]^{1/q} \|v\|_q + K^r |\Omega'|^{1/\chi q},$$

that is, $\|v\|_{\chi q} \leq D_1 \|v\|_q + D_2$, where

$$D_1 = C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{r^q}{K^q} \right]^{1/q} \quad \text{and} \quad D_2 = K^r |\Omega'|^{1/\chi q}.$$

Since v depends on r and ℓ , we name it by $v_{r,\ell}$. In the same way, $D_1 = D_1(r)$ and $D_2 = D_2(r)$. Hence, the last inequality can be rewritten as

$$\|v_{r,\ell}\|_{\chi q} \leq D_1(r) \|v_{r,\ell}\|_q + D_2(r). \quad (8.5)$$

Taking $r = 1$, we have $v = u + K$ and, then

$$\|u + K\|_{\chi q} \leq D_1(1) \|u + K\|_q + D_2(1).$$

Hence $u + K \in L^{\chi q}$ and, therefore, $(u + K)^\chi \in L^q$. Taking $r = \chi$, we have $|v_{\chi,\ell}| \leq (u + K)^\chi$ for any ℓ . Thus $\|v_{\chi,\ell}\|_q \leq \|(u + K)^\chi\|_q$ and, from (8.5),

$$\|v_{\chi,\ell}\|_{\chi q} \leq D_1(\chi) \|(u + K)^\chi\|_q + D_2(\chi).$$

Using that $v_{\chi,\ell} \uparrow (u + K)^\chi$ as $\ell \rightarrow \infty$, we get

$$\|(u + K)^\chi\|_{\chi q} \leq D_1(\chi) \|(u + K)^\chi\|_q + D_2(\chi).$$

Therefore, $u + K \in L^{\chi^2 q}$. More generally, if we take $r = \chi^n$, it follows in a similar way that

$$\|(u + K)^{\chi^n}\|_{\chi q} \leq D_1(\chi^n) \|(u + K)^{\chi^n}\|_q + D_2(\chi^n)$$

and $u + K \in L^{\chi^{n+1} q}$. Thus $u + K$ is an L^r function for any $r \geq 1$. Hence, making $\ell \rightarrow \infty$ in (8.5), we get

$$\|u + K\|_{r\chi q}^r \leq D_1(r) \|u + K\|_{rq}^r + D_2(r).$$

Observe now that $D_1(r) = rH$, where

$$H = C_0 \left[\frac{1}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{1}{K^q} \right]^{1/q}.$$

Furthermore,

$$D_2(r) = K^r |\Omega'|^{1/\chi q} \leq \frac{\|u + K\|_{rq}^r}{|\Omega'|^{1/q}} |\Omega'|^{1/\chi q} \leq r \|u + K\|_{rq}^r |\Omega'|^{(1/\chi - 1)1/q}.$$

Therefore, the last three relations imply

$$\|u + K\|_{r\chi q} \leq r^{1/r} H_0^{1/r} \|u + K\|_{rq}, \quad (8.6)$$

for $r \geq 1$ and $\chi = n/(n-q)$, where $H_0 = H + |\Omega'|^{(1/\chi-1)/q}$. Taking $r = \chi^m$ in (8.6), we have

$$\|u + K\|_{\chi^{m+1}q} \leq \chi^{m/\chi^m} H_0^{1/\chi^m} \|u + K\|_{\chi^m q} \quad \text{for } m \in \mathbb{N} \cup \{0\}.$$

Hence, defining $A_m = \sum_{j=0}^m j/\chi^j$ and $B_m = \sum_{j=0}^m 1/\chi^j$, it follows that

$$\|u + K\|_{\chi^{m+1}q} \leq \chi^{A_m} H_0^{B_m} \|u + K\|_q \quad \text{for } m \in \mathbb{N} \cup \{0\}.$$

Since A_m and B_m are convergent series,

$$\sup(u + K) \leq \chi^A H_0^B \|u + K\|_q,$$

where $A = \lim_{m \rightarrow \infty} A_m$ and $B = \lim_{m \rightarrow \infty} B_m = \frac{\chi}{\chi-1}$. Then

$$\sup u \leq D(H^B + |\Omega'|^{B(1/\chi q - 1/q)})(\|u\|_q + \|K\|_q),$$

for $D = \chi^A 2^B$. Observe that $B(\frac{1}{\chi q} - \frac{1}{q}) = -\frac{1}{q}$. Therefore

$$\sup u \leq D(H^B + |\Omega'|^{-1/q})(\|u\|_q + K|\Omega'|^{1/q}).$$

Notice that

$$H \leq C_0 2^{2/q} \left(\frac{\alpha}{C_*} + \frac{\beta}{C_*} + 1 \right) \left(1 + \frac{1}{K} \right).$$

Then, taking $K = |\Omega'|^{1/n}$, it follows that

$$H^B \leq C_1 (|\Omega'|^{1/n} + 1)^B |\Omega'|^{-1/q},$$

where $C_1 = [C_0 2^{2/q} (\alpha/C_* + \beta/C_* + 1)]^B$. Hence

$$\begin{aligned} \sup u &\leq 2DC_1 \left(|\Omega'|^{1/n} + 1 \right)^B |\Omega'|^{-1/q} (\|u\|_q + K|\Omega'|^{1/q}) \\ &\leq C(|\Omega'|^{1/n} + 1)^B \left(|\Omega'|^{-1/q} \|u\|_q + |\Omega'|^{1/n} \right), \end{aligned}$$

proving the result.

Case 3: $q = n$: Taking $\tilde{q} < q = n$, we get the same estimate as in (8.3) with \tilde{q}^* instead of q^* . Hence

$$\|v - K^r\|_{\tilde{q}^*} \leq C_0 \|\nabla v\|_{\tilde{q}},$$

where $\tilde{q}^* = n\tilde{q}/(n - \tilde{q})$ and $C_0 = \frac{\tilde{q}(n-1)}{n-\tilde{q}}$. Therefore, from Hölder inequality,

$$\|v - K^r\|_{\tilde{q}^*} \leq C_0 \|\nabla v\|_q |\Omega'|^{(q-\tilde{q})/q\tilde{q}}.$$

For $\tilde{q} > n/2$ we get $\tilde{q}/(n - \tilde{q}) > 1$ and, then, $\tilde{q}^* > n = q$. In this case,

$$\|v - K^r\|_{\chi q} \leq C_0 \|\nabla v\|_q |\Omega'|^{(q-\tilde{q})/q\tilde{q}},$$

where $\chi = \tilde{q}^*/q > 1$. Using this and (8.2), it follows that

$$\|v - K^r\|_{\chi q} \leq C_0 \left[\frac{r^q}{C_*} \cdot \left(\alpha + \frac{\beta}{K^{q-1}} \right) + \frac{r^q}{K^q} \right]^{1/q} \|v\|_q |\Omega'|^{(q-\tilde{q})/q\tilde{q}}.$$

This estimate is basically the same as in (8.4). Hence, taking $K = |\Omega|^{1/n}$ and following the same argument as before we get the result. \square

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